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# **BULETINUL INSTITUTULUI POLITEHNIC DIN IAȘI**

**Tomul XXV (XXIX)**

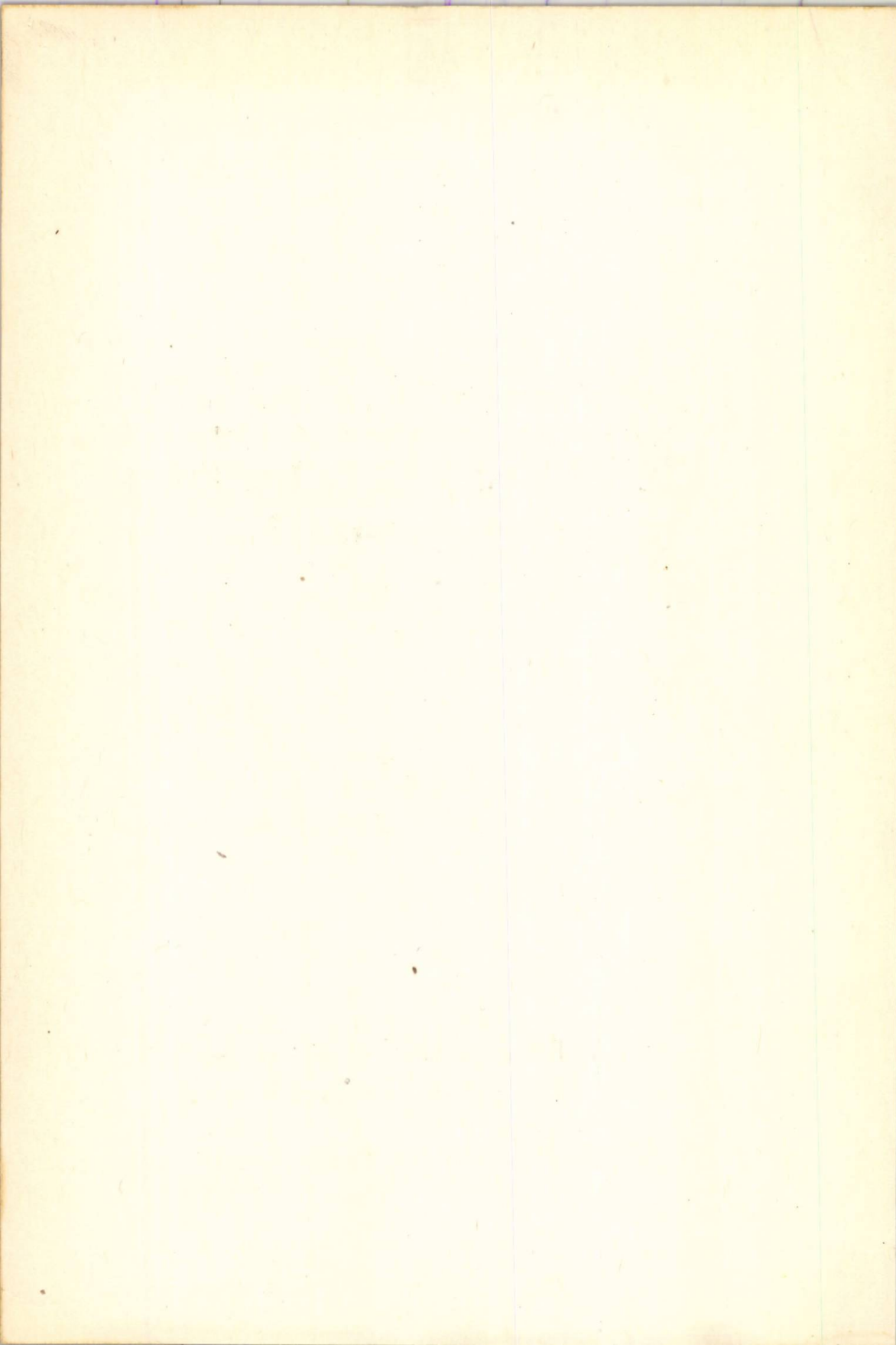
**Fasc. 1 – 2**

**MATEMATICĂ**  
**SECȚIA I MECANICĂ TEORETICĂ**  
**FIZICĂ**

1979

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**MATEMATICĂ**  
**SECȚIA I MECANICĂ TEORETICĂ**  
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**1979**

1989

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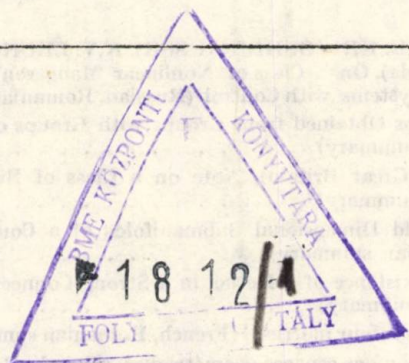
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# ИССЛЕДОВАНИЕ НЕЛИНЕЙНЫХ ПОЛИВОЛНОВЫХ СИСТЕМ МАНЖЕРОНА С НАСЛЕДСТВЕННОСТЬЮ И АВТОРЕГУЛИРОВАНИЕМ В ПРЕДЕЛАХ ИНТЕГРИРОВАНИЯ (II)

Д.Д. БАИНОВ, Л.Е. КРИВОШЕИН, Н. В. ЛЕН и М.Н. ОГЮЗТОРЕЛИ

*Посвящается Профессору  
Д. Манжерону к его юбилею*

1. Непрестанное расширение областей приложений поливолновых уравнений Манжерона [1] — [4], характеризуемых присутствием дифференциальных операторов вида  $M_m^{(n)} \equiv \partial^{mn} / \partial x_1^{(n)} \partial x_2^{(n)} \dots \partial x_m^{(n)}$  известных под именем „полных производных“ корифея науки М. П и к о н е [5] — [6], как напр. автоматическое проектирование поверхностей любой формы [7] — [8], исследование некоторых ядерных процессов [9], определение решений таких уравнений при разнообразных граничных условиях [10], параллельно с возможностями обобщений и.-д. систем с гиперболическими операторами, моделирующих в последнее время физические явления, как напр. электромагнитные свойства диэлектриков [11], импулсировано авторов продолжить их и их сотр. исследования [12] касающиеся теперь ниже выкристаллизованному новому классу поливолновых систем.

2. Пусть

$$(1) \quad M_2^{(2)}[u(x, t)] \equiv \frac{\partial^4 u(x, t)}{\partial x^2 \partial t^2} =$$

$$= f \left[ x, t, u(x, t), M_2^{(1)}[u(x, t)], \int_{s(x, M_2^{(1)}[u])}^{r(x, M_2^{(1)}[u])} K(x, t, \xi, u(\xi, t), M_2^{(1)}[u(\xi, t)]) d\xi \right]$$

новый класс нелинейных поливолновых и.-д. у. Манжерона с авторегулированием в пределах интегрирования, причем, в целях корректного формулирования в смысле Адамамара рассматриваемой задачи, неизвестная функция  $u(x, t)$  удовлетворяет дополнительным граничным условиям

$$(2) \quad u_i^{(i)}(x, \alpha) = \varphi_i \left[ x, \iint_D P(\xi, \tau, M_2^{(1)}[u]) dD \right],$$

$$u_x^{(i)}(a, t) = \Psi_i \left[ t, \iint_D Q(\xi, \tau, M_2^{(1)}[u]) dD \right], \quad i = 0, 1,$$



$$(3) \quad \varphi_{ix}^{(i)}[\cdot]_{x=a} = \psi_{it}^{(i)}[\cdot]_{t=\alpha} = q_i = \text{const}, \quad i = 0, 1, \quad D = [a, b] \times [\alpha, \gamma],$$

где

$$s(x, M_2^{(1)}[u]) = s \left[ x, \int_{\alpha}^{\gamma} N(x, \tau, M_2^{(1)}[u(x, \tau)]) d\tau \right],$$

$$r(x, M_2^{(1)}[u]) \equiv r \left[ x, \int_{\alpha}^{\gamma} R(x, \tau, M_2^{(1)}[u]) d\tau \right],$$

функции  $f(x, t, l_1, l_2, l_3)$ ,  $K(x, t, \xi, l_4, l_5)$ ,  $N(x, \tau, l_7)$ ,  $R(x, \tau, l_3)$ ,  $s(x, l_6)$ ,  $r(x, l_8)$ ,  $\varphi_i(x, l_{10})$ ,  $P(\xi, \tau, l_{11})$ ,  $\psi_i(t, l_{12})$  и  $Q(\xi, \tau, l_{11})$  определены, непрерывны и по  $l_i$ ,  $i = \overline{1, 12}$ , удовлетворяют условию Липшица с непрерывными и неотрицательными коэффициентами, соответственно,

$$L_{if}(x, t), \quad i = 1, 2, 3, \quad L_{iK}(x, t, \xi), \quad i = 1, 2, \quad L_s(x), \quad L_N(x, \tau), \\ L_r(x), \quad L_R(x, \tau), \quad L_{\varphi_i}(x), \quad L_P(\xi, \tau), \quad L_{\psi_i}(t), \quad L_Q(\xi, \tau), \quad i = 0, 1, \quad \text{в области}$$

$\{D_1 = a \leq \xi, s(x, l_6), r(x, l_8), x \leq b, \alpha \leq \tau, t \leq \gamma, 0 \leq |l_i| \leq r_i, i = \overline{1, 14}\}$   
 $0 \leq |\varphi_0(x, l_{10})|, |\psi_0(t, l_{12})| \leq r_1, 0 \leq |\varphi_1(x, l_{10})| \leq r_{13}, 0 \leq |\psi_1(t, l_{12})| \leq r_{12}$ ,  
 функции  $\varphi_i(x, l_{10})$ ,  $\psi_i(t, l_{12})$ , кроме того, дважды непрерывно дифференцируемы по первым аргументам, а  $L_{\varphi_i}(x)$ ,  $L_{\psi_i}(x)$  — коэффициенты Липшица функций  $\varphi'_{ix}(x, l_{10})$ ,  $\psi'_{it}(x, l_{12})$  по вторым аргументам.

3. Проблема разрешимости системы (1) — (3). Воспользуясь уравнение

$$(4) \quad M_2^{(1)}[u(x, t)] = \varphi'_{ix} \left[ x, \iint_D P(\xi, \tau, M_2^{(1)}[u]) dD \right] + \\ + \psi'_{it} \left[ t, \iint_D Q(\xi, \tau, M_2^{(1)}[u]) dD \right] - q_1 + \iint_{\alpha\alpha} f[\eta, \theta, u(\eta, \theta),$$

$$M_2^{(1)}[u(\eta, \theta)], \int_{s(\eta, M_2^{(1)}[u])}^{r(\eta, M_2^{(1)}[u])} K(\eta, \theta, \xi, u(\xi, \theta), M_2^{(1)}[u(\xi, \theta)]) d\xi \Big] d\theta d\eta,$$

с граничными условиями (2) — (3) при  $i = 0$ , приходим к следующей, эквивалентной с рассматриваемой проблемой, системе уравнений

$$(5) \quad u(x, t) = \varphi_0 \left[ x, \iint_D P(\xi, \tau, v) dD \right] + \psi_0 \left[ t, \iint_D Q(\xi, \tau, v) dD \right] - q_0 + \\ + \iint_{\alpha\alpha} v(\xi, \tau) d\tau d\xi \equiv A[v],$$



$$(6) \quad v(x, t) = \varphi'_{1x} \left[ x, \iint_D P(\xi, \tau, v) dD \right] + \psi'_{1t} \left[ t, \iint_D Q(\xi, \tau, v) dD \right] - q_1 + \\ + \iint_{\alpha x}^{xt} f \left[ \eta, \theta, A[v], v(\eta, \theta), \int_{s(\eta, v)}^{r(\eta, v)} K(\eta, \theta, \xi, A[v], v(\xi, \theta)) d\xi \right] d\theta d\eta = H[v].$$

Пусть операторы  $A[.]$  и  $H[.]$  отображают область  $D_1$  на себя,  $\|.\| = \max_{D_1} |.|$  и выполняется неравенство

$$(7) \quad k = \left\| L_{\varphi'_{1x}}(x) \iint_D L_P(\xi, \tau) dD + L_{\psi'_{1t}}(t) \iint_D L_Q(\xi, \tau) dD + \right. \\ + \iint_{\alpha x}^{xt} \left\{ L_{1f}(\eta, \theta) g(\eta, \theta) + L_{2f}(\eta, \theta) + L_{3f}(\eta, \theta) \left[ \left( L_s(\eta) \int_{\alpha}^{\gamma} L_N(\eta, \tau) d\tau + \right. \right. \right. \\ \left. \left. \left. + L_r(\eta) \int_{\alpha}^{\gamma} L_A(\eta, \tau) d\tau \right) \|K(\cdot)\| + \left| \int_{s(\eta, v_*)}^{r(\eta, v_*)} [L_{1K}(\eta, \theta, \xi) g(\xi, \theta) + \right. \right. \right. \\ \left. \left. \left. + L_{2K}(\eta, \theta, \xi)] d\xi \right| \right] \right\} d\theta d\eta \left\| < 1, \quad \forall v_* \in D_1,$$

где

$$g(x, t) \equiv L_{\varphi_0}(x) \iint_D L_P(\xi, \tau) dD + L_{\psi_0}(t) \iint_D L_Q(\xi, \tau) dD + (x - a)(t - \alpha).$$

Отсюда вытекает следующая

**Теорема 1.** При условии отображения операторами  $A[.]$  и  $H[.]$  области  $D_1$  на себя и выполнении неравенства (7), нелинейная поливолновая и.-д. система (1) – (3) имеет в области  $D_1$  единственное решение  $u = u(x, t) \in C^{2,2} [a, b] \times [\alpha, \gamma]$ , которое представляется равенством (5), а  $v(x, t) \in C^{1,1} [a, b] \times [\alpha, \gamma]$  единственное решение уравнения (6), причем эти решения являются пределами последовательностей  $\{u_i(x, t)\}_0^\infty$ ,  $\{v_i(x, t)\}_0^\infty$ :

$$(8) \quad u_i(x, t) = A[v_i], \quad i = 0, 1, \dots,$$

$$(9) \quad v_i(x, t) = H[v_{i-1}], \quad i = 1, 2, \dots, \quad \forall(x, t), \quad v_0 \in D_1,$$

соответствующая погрешность при остановке на  $i$ -ом шаге будучи дана неравенством

$$(10) \quad |u(x, t) - u_i(x, t)| \leq g(x, t) k^i \|v_0 - H[v_0]\| : (1 - k) \xrightarrow{i \rightarrow \infty} 0.$$

В самом деле, при условии выполнения вышеупомянутых свойств нелинейных операторов  $A[.]$  и  $H[.]$  имеет место принцип Банаха или же

действительны эквивалентны ему теоремы о неподвижных точках в соответствующих функциональных пространствах.

4. Имея в виду все расширяемую область задач Гурса [13], рассмотрим теперь для класса и.-д. у. (1) такую задачу

$$(11) \quad u_i^{(i)}(x, \alpha) = \varphi_i(x), \quad u_x^{(i)}(u, t) = \psi_i(t), \quad i = 0, 1,$$

при исполнении условий (3). Следуя методу части из авторов и их сотрудников выработанному в [14], задача (1), (3), (11) сводится к эквивалентной ей системе интегральных уравнений

$$(12) \quad u(x, t) = \varphi_0(x) + \psi_0(t) - q_0 + [\psi_1(t) - \psi_1(\alpha)](x - a) + [\varphi_1(x) - \varphi_1(a)](t - \alpha) - q_1(x - a)(t - \alpha) + \iint_{\alpha\alpha}^{xt} (x - \eta)(t - \theta) v(\eta, \theta) d\theta d\eta \equiv A_0[v],$$

$$(13) \quad v(x, t) = f\left[x, t, A_0[a], M_2^{(1)}[A_0[v]], \int_{s(\eta, M_2^{(1)}[A_0[v]])}^{r(\eta, M_2^{(1)}[A_0[v]])} K(x, t, \xi, A_0[v]), \right.$$

$$\left. M_2^{(1)}[A_0[v]] d\xi \right] \equiv H_0[v].$$

Таким образом, если операторы  $A_0[.]$  и  $H_0[.]$  отображают область  $D_1$  на себя и имеет место неравенство

$$(14) \quad k_0 = \|A_0\| < 1,$$

то задача (1), (3), (11) имеет в области  $D_1$  единственное решение  $u = u(x, t) \in C^{2,2}[a, b] \times [\alpha, \gamma]$  к которому сходится абсолютно и равномерно функциональная последовательность  $\{u_i(x, t)\}_{i=0}^{\infty}$ ,

$$(15) \quad u_i(x, t) = A_0[v_i], \quad i = 0, 1, \dots;$$

$$(16) \quad v_i(x, t) = H_0[v_{i-1}], \quad i = 1, 2, \dots \quad \forall (x, t), v_0 \in D$$

и при этом оценка сходимости выражается неравенством,

$$(17) \quad |u(x, t) - u_i(x, t)| \leq [(x - a)(t - \alpha) : 2]^2 k_0^i \|v_0 - H_0[v_0]\| : (1 - k_0) \xrightarrow{i \rightarrow \infty} 0.$$

5. Задача устойчивости Решений поливолновой и.-д. системы (1) — (3). Рассмотрим теперь вопрос о непрерывной зависимости решения задачи (1) — (3) от функций  $P(\xi, \tau, l_{11})$  и  $Q(\xi, \tau, l_{11})$ . Пусть эти функции, после воздействия на них некоторого возмущения, переходят, соответственно, в функции  $P_1(\xi, \tau, l_{13})$ ,  $Q_1(\xi, \tau, l_{13})$ , определенные и непрерывные в области  $D_1$ , удовлетворяющие по  $l_{13}$  условию Липшица с непрерывными и неотрицательными коэффициентами, соответственно,  $L_{P_1}(\xi, \tau)$ ,  $L_{Q_1}(\xi, \tau)$  и, кроме того,  $|P(\xi, \tau, l_{11}) - P_1(\xi, \tau, l_{13})| \leq d_0(\xi, \tau) + d_1(\xi, \tau)|l_{11} - l_{13}|$ ,  $|Q(\xi, \tau, l_{11}) - Q_1(\xi, \tau, l_{13})| \leq d_2(\xi, \tau) + d_3(\xi, \tau)|l_{11} - l_{13}|$ , где  $d_i(\xi, \tau)$ ,  $i = 0, 3$  — определенные, непрерывные и неотрицательные функции в  $D_1$ ;  $0 \leq |l_{13}| \leq r_{11}$ .



В таком случае (2), (3) переходит в

$$(18) \quad W_i^{(4)}(x, \alpha) = \varphi_i \left[ x, \iint_D P_1(\xi, \tau, M_2^{(1)}[w]) dD \right];$$

$$w_x^{(4)}(a, t) = \psi_i \left[ t, \iint_D Q_1(\xi, \tau, M_2^{(1)}[w]) dD \right], \quad i = 0, 1,$$

$$(19) \quad \varphi_{ix}^{(4)}[.]_{x=a} = \psi_{it}^{(4)}[.]_{t=\alpha} = q_{1i}, \quad i = 0, 1,$$

причем новая неизвестная функция  $w(x, t)$  удовлетворяет уравнению аналогичному уравнению (1) — назовем его  $(1^*)$  — в котором функция  $u(x, t)$  заманена  $w(x, t)$ . Поступая с системой  $(1^*)$ , (18), (19) так же как и с задачей (1) — (3), приходим к системе интегральных уравнений

$$(20) \quad w(x, t) = A_1[z], \quad (21) \quad z(x, t) = H_1[z].$$

Откуда следует

**Т е о р е м а 2.** *Нелинейная поливолновая и.-д. система  $(1^*)$ , (18), (19) с авторегулированием имеет единственное решение в области  $D_1$ ,  $w = w(x, t) \in C^{2,2}[a, b] \times [\alpha, \gamma]$ , которое можно построить как предел последовательности  $\{w_i(x, t)\}_0^\infty$ , где*

$$(22) \quad w_i(x, t) = A_1[z_i], \quad i = 0, 1, \dots,$$

$$(23) \quad z_i(x, t) = H_1[z_{i-1}], \quad i = 1, 2, \dots, \quad \forall(x, t), \quad z_0 \in D$$

как только операторы  $A_1[.]$  и  $H_1[.]$  отображают область  $D_1$  на себя и

$$(24) \quad k_1 = \|A_1\| < 1$$

а при этом допускаемая погрешность, приняв за приближенное решение  $i$ -го шага  $w_i(x, t)$ , оценивается неравенством

$$(25) \quad |w(x, t) - w_i(x, t)| \leq g_1(x, t) k_1^i \|z_0(x, t) - H_1[z_0]\| : (1 - k_1) \xrightarrow{i \rightarrow \infty} 0,$$

где

$$g_1(x, t) \equiv L_{\varphi_0}(x) \iint_D L_{P_1}(\xi, \tau) dD + L_{\psi_0}(t) \iint_D L_{Q_1}(\xi, \tau) dD + (x - a)(t - \alpha).$$

6. Оценка близости решений исходной и возмущенной задачи выражается следующим неравенством

$$(26) \quad |u(x, t) - w(x, t)| \equiv |A[v] - A_1[z]| \leq |q_0 - q_{10}| + L_{\varphi_0}(x) \iint_D d_0(\xi, \tau) dD +$$

$$+ L_{\psi_0}(t) \iint_D d_2(\xi, \tau) dD + \left[ L_{\varphi_0}(x) \iint_D d_1(\xi, \tau) dD + L_{\psi_0}(t) \iint_D d_3(\xi, \tau) dD + \right.$$

$$\left. + (x - a)(t - \alpha) \right] \|v - z\| \equiv I_1(x, t) + I_2(x, t) \|v - z\| \leq$$

$$\leq I_1(x, t) + I_2(x, t) \|z_0 - A[z_0]\| : (1 - k).$$



*Примечание.* В одной из принадлежащей этой же серии работ авторов будет рассмотрена корректность задачи решения другого класса нелинейных поливолновых и.д.у. Манжерона и изучена устойчивость ее решений при некоторых пертурбациях управления, причем изложенный там метод распространяем на аналогичные системы с полными производными Пиконе более высокого порядка, а в последующей работе будет углублена связь с серией недавних работ первого из авторов и его сотрудников [15].

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STUDII PRIVIND SISTEMELE POLIVIBRANTE MANGERON CU OPERATORI  
EREDITARI ȘI CONTROL ÎN LIMITELE DE INTEGRARE (II)

(Rezumat)

Autorii, subliniind lărgirea actuală a domeniilor de aplicație a ecuațiilor polivibrante [7]—[12] stabilesc o serie de teoreme privind o nouă clasă de sisteme integro-diferențiale polivibrante neliniare cu autoreglare în limitele de integrare.





# ON GROUPS OBTAINED FROM GROUPS WITH GROUPS OF OPERATORS

BY

GH. COSTOVICI

1. In this paper we derive groups from groups with groups of operators by means of homomorphisms. We establish some properties of them and give some examples.

2. If  $V$  is an additive group with the multiplicative group  $K$  as operator domain operating on the right so that  $\forall v_1, v_2, v \in V$  and  $\forall k_1, k_2, k \in K$  we have  $(v_1 + v_2)k = v_1k + v_2k$ ,  $(vk_1)k_2 = v(k_1k_2)$  and  $v1 = v$  ( $1$  is the unity of  $K$ ), then  $V$  will be called, for short,  $\bar{K}$ -group. We note that if  $0$  is the unity of  $V$  then from the first axiom follows  $0k = 0$  and  $(-v)k = -(vk)$  and from the third,  $VK = V$ . Also  $vk = 0 \Rightarrow v = 0$ .

3. Proposition 1. If  $V_i$  are  $\bar{K}_i$ -groups,  $i = \overline{1, n}$ ,  $n \geq 2$ , and  $f_i$  are homomorphisms from  $V_i$  into  $K_{i+1}$ ,  $i = \overline{1, n^2}$ , and if  $f_{i+1}(v_{i+1}f_i(v_i)) = f_{i+1}(v_{i+1}) \forall v_{i+1} \in v_{i+1}$  and  $\forall v_i \in V_i$ ,  $i = \overline{1, n}$ , then the set  $\prod_0 V_i = \{(v_1, v_2, \dots, v_n) = (v_{i+2}); v_i \in V_i, i = \overline{1, n}\}$  becomes group by the operation  $(v_{i+2}) \circ (v'_{i+2}) = (v_{i+2}f_{i+1})(v'_{i+1}) + v'_{i+2}$ . It will be noted by  $\prod_0 V_i$ .

For the proof we specify only that if  $0_i$  is the unity of  $V_i$ ,  $i = \overline{1, n}$ , then the unity of  $\prod_0 V_i$  is  $(0_i)$  and that the inverse of  $(v_{i+1})$  in  $\prod_0 V_i$  is  $(-v_{i+1}f_i(-v_i))$ .

We note that more properties of  $\prod_0 V_i$  and more examples are in [1].

Proposition 2. If the assumptions of the Proposition 1 are fulfilled and if  $V_{j-1}$  and  $V_j$  ( $j$  fixed among  $i = \overline{1, n}$  are commutative, then the set  $\prod_0 V_1$  becomes group by the operation  $(v_j \circ_j (v'_i)) = (v_1f_n(v'_n) + v'_1, v_2f_1(v'_1) + v'_2, \dots, v_{j-1}f_{j-2}(v'_{j-2}) + v'_{j-1}, v'_jf_{j-1}(v_{j-1}) + v_j, v_{j+1}f_j(v'_j) + v'_{j+1}, \dots, v_nf_{n-1}(v'_{n-1}) + v'_n)$ . It will be noted by  $\prod_{0,j} V_i$ .

We make the remark that  $0_j$  differs from  $0$  only at the  $j$  component.

For the proof we specify that the unity and the inverse are the same as in  $\prod_0 V_i$ .

1) In this paper an index  $i$  should always read  $i(\text{mod } n)$ .



At this point we note that there can exist groups on the same set for which: a) the unities and inverses are different; b) the unities coincide and the inverses do not; c) the inverses coincide and the unities do not (in a Boolean ring and its dual); d) the unities coincide and so do the inverses.

It can be proved the Propositions 3, 4, 5, 6 below.

**Proposition 3.** *If the hypotheses of the Proposition 2 are fulfilled and in addition,  $K_i$  are commutative and  $f_i$  are surjections,  $i = \overline{1, n}$ , and if  $\prod_1^n K_i$  is usual direct product of groups, then,  $\prod_0 V_i$  and  $\prod_{0_j} V_i$  are  $\prod_1^n K_i$ -groups by the external operation  $(v_i)(k_i) = (v_i k_i)$ .*

**Proposition 4.** *If the assumptions of the Proposition 2 are fulfilled and if  $V_i$  are topological groups,  $i = \overline{1, n}$ ,  $K_i$  are topological spaces,  $i = \overline{1, n}$ , and all external operations and all  $f_i$  are continuous, then,  $\prod_0 V_i$  and  $\prod_{0_j} V_i$  are topological groups by the product topology.*

**Proposition 5.** *If the hypotheses of the Proposition 3 are fulfilled and in addition,  $f_{j-1}(v_{j-1}) = f_{j-1}(-v_{j-1}) \forall v_{j-1} \in V_{j-1}$ ,  $j$  fixed among  $i = \overline{1, n}$ , then, the mapping  $F_j: \prod_0 V_i \rightarrow \prod_{0_j} V_i$  defined by  $F_j(v_1, \dots, v_n) = (v_1, v_2, \dots, v_{j-1}, v_j f_{j-1}(v_{j-1}), v_{j+1}, \dots, v_n)$  is isomorphism of groups with operators.*

**Corollary.** *If the hypotheses of the Proposition 5 are fulfilled and if  $V_i$ ,  $i = \overline{1, n}$  and  $K_j$  ( $j$  fixed) are topological spaces and  $f_{j-1}$  and the external operation in  $V_j$  are continuous, then,  $F_j$  is isomorphism of groups with operators and homeomorphism if  $\prod_0 V_i$  and  $\prod_{0_j} V_i$  are equipped with the product topology.*

**Proposition 6.** *If  $V$  is  $K$ -group and if  $f$  is homomorphism from  $V$  to  $K$  so that  $\forall a, b \in V f(ab) = f(a)$ , then  $V$  becomes group under the operation  $v_1 \circ v_2 = v_1 f(v_2) + v_2$ . It will be noted by  $\prod_0 V$ .  $V$  has the properties:*

1)  $K$  commutative and  $f$  and surjection  $\Rightarrow V = K$ -group by the same external operation.

2)  $V$  topological group,  $K$  topological space and  $f$  external operation continuous  $\Rightarrow V$  is topological group by the same topology.

4. In the examples that follow we take into account that any commutative group  $G$  can be regarded as  $U$ -group,  $U$  = the multiplicative group  $\{ -1, 1 \}$ , where  $g1 = g$  and  $g(-1) = g^{-1}$ . If  $f$  is homomorphism from  $G$  to  $U$ , then  $f(g) = f(g^{-1})$  for  $f(gg) = f(g)f(g) = 1 \Rightarrow f(g) = f(g^{-1})$ .

**Direct examples.** One can apply straight the Propositions 5 and 6 to the case  $V_i = \mathbb{Z}$  = the additive group of all integers,  $K_i = U$ ,  $f_i(z) = (-1)^z$ ,  $i = \overline{1, n}$ .

**Indirect examples.** I. Let  $G_i$  be multiplicative and commutative group and let  $f_i$  be homomorphism from  $G_i$  to  $U$ ,  $i = \overline{1, n}$ ,  $n \geq 2$ . Let  $n_j$  be integers so that  $1 \leq n_j \leq n$ .  $j = \overline{1, k}$ ,  $k \geq 2$ ,  $n_1 + n_2 + \dots + n_k = n$ . We note  $n_{sj} = n_1 + n_2 + \dots + n_j$ ,  $j = \overline{1, k}$ , and consider the usual direct product



of groups,  $\prod_1^n U = U \times U \times \dots \times U$  ( $n$  times). Then the set  $\prod_1^n G_i$  becomes  $\prod_1^n U$ -group by every one of the internal operations  $0$  and  $0_j$  so defined:

$$\begin{aligned}
 (g_i) \circ (g'_i) &= \left( g_1 \prod_{n_{sk-1}+1}^{n_{sk}} f_i(g'_i) \cdot g'_1, g_2 \prod_{n_{sk-1}+1}^{n_{sk}} f_i(g'_i) \cdot g'_2, \dots \right. \\
 &g_{n_1} \prod_{n_{sk-1}+1}^{n_{sk}} f_i(g'_i) \cdot g'_{n_1}, g_{n_1+1} \prod_1^{n_{s1}} f_i(g'_i) \cdot g'_{n_1+1}, \dots, g_{n_1+n_2} \prod_1^{n_{s1}} f_i(g'_i) \cdot g'_{n_1+n_2}, \\
 &\dots, g_{n_{sk-1}+1} \prod_{n_{sk-2}+1}^{n_{sk-1}} f_i(g'_i) \cdot g'_{n_{sk-1}+1}, \dots, g_n \prod_{n_{sk-2}+1}^{n_{sk-1}} f_i(g'_i) \cdot g'_n \Big), \\
 (g_i) \circ_j (g'_i) &= \left( g_1 \prod_{n_{sk-1}+1}^{n_{sk}} f_i(g'_i) \cdot g'_1, \dots, g_{n_1} \prod_{n_{sk-1}+1}^{n_{sk}} f_i(g'_i) \cdot g'_{n_1}, \right. \\
 &g_{n_1+1} \prod_1^{n_1} f_i(g'_i) \cdot g'_{n_1+1}, \dots, g_{n_1+n_2} \prod_1^{n_1} f_i(g'_i) \cdot g'_{n_1+n_2}, \dots, \\
 &g_{n_{sj}+1} \prod_{n_{sj-2}+1}^{n_{sj-1}} f_i(g'_i) \cdot g'_{n_{sj}+1}, \dots, g_{n_{sj}} \prod_{n_{sj-2}+1}^{n_{sj-1}} f_i(g'_i) \cdot g'_{n_{sj}} \cdot g'_{n_{sj}+1}, \\
 &\prod_{n_{sj-1}+1}^{n_{sj}} f_i(g'_i) \cdot g'_{n_{sj}+1}, \dots, g'_{n_{sj}+1} \prod_{n_{sj-1}+1}^{n_{sj}} f_i(g'_i) \cdot g'_{n_{sj}+1}, \\
 &g_{n_{sj}+1} \prod_{n_{sj}+1}^{n_{sj}+1} f_i(g'_i) \cdot g'_{n_{sj}+1}, \dots, g_{n_{sj}+2} \prod_{n_{sj}+1}^{n_{sj}+1} f_i(g'_i) \cdot g'_{n_{sj}+2}, \dots, \\
 &g_{n_{sk-1}+1} \prod_{n_{sk-2}+1}^{n_{sk-1}} f_i(g'_i) \cdot g'_{n_{sk-1}+1}, \dots, g_{n_{sk}} \prod_{n_{sk-2}+1}^{n_{sk-1}} f_i(g'_i) \cdot g'_{n_{sk}} \Big),
 \end{aligned}$$

and by the external operation  $(g_i)(k_i) = (g_i k_i)$ . They will be noted by  $\prod_0 G_i$  and  $\prod_{0_j} G_i$  respectively. The mapping  $F_j: \prod_0 G_i \rightarrow \prod_{0_j} G_i$  where

$$\begin{aligned}
 F_j((g_i)) &= \left( g_1, \dots, g_{n_{sj}}, g_{n_{sj}+1} \prod_{n_{sj-1}+1}^{n_{sj}} f_i(g_i), g_{n_{sj}+2} \prod_{n_{sj-1}+1}^{n_{sj}} f_i(g_i), \right. \\
 &\dots, g_{n_{sj}+1} \prod_{n_{sj-1}+1}^{n_{sj}} f_i(g_i), g_{n_{sj}+1} \prod_{n_{sj-1}+1}^{n_{sj}} f_i(g_i), \dots, g_n \Big)
 \end{aligned}$$

is isomorphism of groups with operators.

For the proof, it must be considered at beginning the direct products of groups  $G_{n_i} = G_{n_{si-1}+1} \times G_{n_{si-1}+2} \times \dots \times G_{n_{si-1}+n_i}$  ( $n_{s0} = 0$ ) and so on. Then we apply the Propositions 1, 2 and 5. What is linked with the external operations must be verified directly.

II. If  $G_i$  are multiplicative and commutative groups and  $f_i$  are homomorphisms from  $G_i$  to  $U$ ,  $i = \overline{1, n}$ , then the set  $\prod_1^n G_i$  becomes group by the operation  $(g_i) \circ (g'_i) = (g_i \prod_1^n f_i(g'_i) \cdot g'_i)$ . It will be noted by  $\prod_0^n G_i$ . For the proof one considers direct product of groups  $G = \prod_1^n G_i$  and the homomorphism  $f: G \rightarrow U$  defined by  $f((g_i)) = \prod_1^n f_i(g_i)$  and one applies the Proposition 6.

One verifies directly that  $\prod_0^n G_i$  becomes  $\prod_1^n U$ -group by the external operation  $(g_i)(k_i) = (g_i k_i)$ .

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#### DESPRE GRUPURI OBTINUTE DIN GRUPURI CU GRUPURI DE OPERATORI

(Rezumat)

Din grupuri cu grupuri de operatori și cu ajutorul unor homomorfisme cu proprietăți speciale se obțin grupuri noi. Se evidențiază unele proprietăți ale lor și se dau exemple.



## NOTE ON A CLASS OF ORTHOGONAL BIPOLYNOMIALS

BY

KATHLEEN M. URWIN

**1. Introduction.** Orthogonal bipolynomials were first discussed by Arscott [1]. A bipolynomial  $p_n(x, y)$  of degree  $n$  in the two variables  $x, y$ , is a function which, when either of  $x, y$  is held constant, becomes a polynomial of degree  $n$  in the other. A bipolynomial  $p_n(x, y)$  is *symmetric* if  $p_n(x, y) = p_n(y, x)$  and *separable* if it can be written as a product  $p(x)q(y)$ . In what follows we consider symmetric bipolynomials only.

To construct an orthogonal system of bipolynomials we take a weight function  $w(x, y) \geq 0$  on a region  $R$  of the  $x$ - $y$  plane such that

$$\iint_R x^m y^n w(x, y) dx dy$$

exists  $\forall m, n \geq 0$ . Let  $L_w^{(2)}(R)$  be the space of all functions  $f(x, y)$  such that

$$\iint_R \{f(x, y)\}^2 w(x, y) dx dy$$

exists, and define the inner product

$$\langle f, g \rangle = \iint_R f(x, y)g(x, y) w(x, y) dx dy.$$

Assuming  $p_0(x, y) = 1$ , we now look for symmetric bipolynomials  $p_n(x, y)$  such that

$$\langle p_m, p_n \rangle = 0, \quad m \neq n.$$

It transpires that for  $n \geq 1$ ,  $p_n(x, y)$  is not uniquely determined, but is a manifold of dimension  $n + 1$  spanned by the bipolynomials  $\pi_n^m = 0, 1, 2, \dots, n$ , where

$$\pi_n^m = (xy)^{n-m} (x + y)^m + q_{n-1}^m(x, y),$$

$q_{n-1}^m(x, y)$  being a bipolynomial of degree  $n - 1$ . We have

$$\langle \pi_n^m, \pi_n^{m'} \rangle = 0, \quad n \neq n',$$

but

$$\langle \pi_n^m, \pi_n^{m'} \rangle = 0, \quad m \neq m'.$$

However, we can construct an orthogonal set from the set  $\{\pi_n^m\}_{m=0}^n$  by the usual Schmidt orthogonalisation process.

An interesting problem is the number of independent separable bipolynomials of each degree in any given orthogonal system. The most useful systems seem likely to be those which

(a) have the full number,  $n + 1$ , separable bipolynomials of degree  $n$ , for each  $n$ ;

(b) have precisely one separable bipolynomial of each degree.

Some special systems having property (a) are known; see [2]. The purpose of this Note is to list some properties of a particular class of bipolynomials having property (b).

2. Take  $f(x) \geq 0$  on an interval  $I$ , finite or infinite, of the real line such that

$$\int_I f(x) x^n dx$$

exists for all  $n \geq 0$ . Let

$$w(x, y) = f(x)f(y) \quad \text{and} \quad R = I \times I.$$

Then the system of orthogonal bipolynomials with weight function  $w(x, y)$  on  $R$  has precisely one separable bipolynomial of each degree.

We denote the separable bipolynomial of degree  $n$  by  $S_n(x, y)$ , where

$$S_n(x, y) = s_n(x) s_n(y),$$

$s_n(x)$  being a polynomial of degree  $n$ , and we take the coefficient of  $x^n$  in  $s_n(x)$  as unity.

(a) The polynomials  $s_n(x)$  form, of course, an orthogonal system with weight function  $f(x)$  on the interval  $I$ . Thus all the classical systems of orthogonal polynomials are included. In particular we note that all the zeros of  $s_n(x)$  are simple and located in  $I$ .

(b) The bipolynomials  $\pi_n^m$ ,  $m = 0, 1, \dots, n$ , can be formed by a simple algebraic process applied to the set of polynomials  $s_0, s_1, \dots, s_n$ .

Let  $Q_n^m(x, y) = s_n(x) s_{n-m}(y) + s_n(y) s_{n-m}(x)$ ,  $m = 0, 1, 2, \dots, n$ . Then

$$\langle Q_n^m, Q_n^{m'} \rangle = 0, \quad n \neq n',$$

and, incidentally,

$$\langle Q_n^m, Q_n^{m'} \rangle = 0, \quad m \neq m'.$$

To derive the  $\pi_n^m$ , take

$$Q_n^m(x, y) = \pi_n^m(x, y).$$

Then

$$Q_n^{n-1}(x, y) = \pi_n^{n-1}(x, y) + C\pi_n^n(x, y),$$



where  $C$  is a known constant. Thus  $\pi_n^{n-1}(x, y)$  is obtained, and the process is continued to obtain all the  $\pi_n^m$ .

(c) The form of the bipolynomial  $\pi_n^m$ , at least for small values of  $n$  is easily found. For example, if we take

$$s_0(x) = 1, \quad s_1(x) = x + c, \quad s_2(x) = x^2 + dx + e$$

then

$$\pi_0^0(x, y) = 1, \quad \pi_1^1(x, y) = x + y + 2c, \quad \pi_1^0(x, y) = xy - c^2,$$

$$\pi_2^2(x, y) = (x + y)^2 - 2xy + d(x + y) + 2e,$$

$$\pi_2^1(x, y) = xy(x + y) + 2dxy + e(x + y),$$

$$\pi_2^0(x, y) = x^2 y^2 - de(x + y) - d^2 xy - e^2.$$

In the case where  $f(x)$  is even and  $I$  is centred at the origin, considerable simplification occurs in the bipolynomials.

**3. Zeroids.** Since  $p_n(x, y)$  is a function of two variables its zeros lie on a curve, usually with several branches, in the  $x - y$  plane. We shall call the curve given by  $p_n(x, y) = 0$  a *zeroid*, and we shall denote the zeroid corresponding to  $\pi_n^m(x, y)$  by  $Z_n^m$ .

(a) The zeroid corresponding to a separable bipolynomial  $S_n(x, y)$  is a set of  $2n$  straight lines,  $n$  parallel to each axis, these lines being

$$x = x_r, \quad y = y_s, \quad r, s = 1, 2, \dots, n,$$

where

$$s_n(x_r) = 0 = s_n(y_s).$$

(b) Each zeroid  $Z_n^m$ ,  $m = 0, 1, \dots, n$ , though not, of course, each branch of the zeroid, passes through the set of points  $(x_r, y_s)$ ,  $r, s = 1, 2, \dots, n$ . Every zeroid is symmetric about  $y = x$ , and with the exception of the branches  $x = 0$ ,  $y = 0$ , which can occur in some systems, all branches cut  $y = x$  orthogonally. Thus branches through the points  $(x_r, x_r)$ ,  $r = 1, 2, \dots, n$  touch each other there. Some branches of a zeroid may cross  $y = x$  at points in  $R$  other than those of the set  $(x_r, x_r)$ ; see, for example, the zeroid  $Z_4^0$  in the first system quoted below, where there are two such points.

(c) In the systems where  $f(x)$  is even and  $R$  is a square, centre the origin, the lines  $x = 0$ ,  $y = 0$ ,  $x + y = 0$  can appear as branches. All other branches cross  $y + x = 0$  orthogonally and so all such branches through the points  $(x_r, -x_r)$ ,  $r = 1, 2, \dots, n$  touch each other there.

(d) The zeroids rapidly become complicated curves as  $n$  increases, even for the simplest systems. Two simple examples are given below.

#### 4. Simple Cases.

(a)  $w(x, y) = 1$ ,  $R = [-1, 1] \times [-1, 1]$ .

This is the simplest system. The polynomials  $s_n(x)$  are constant multiples of the Legendre polynomials so that

$$s_0(x) = 1, \quad s_1(x) = x, \quad s_2(x) = x^2 - \frac{1}{3}, \quad s_3(x) = x^3 - \frac{3}{5}x, \dots$$

We find

$$\begin{aligned} \pi_1^1 &= x + y, \quad \pi_1^0 = xy, \\ \pi_2^2 &= (x + y)^2 - 2xy - \frac{2}{3}, \quad \pi_2^1 = xy(x + y) - \frac{1}{3}(x + y), \quad \pi_2^0 = x^2y^2 - \frac{1}{9}. \end{aligned}$$

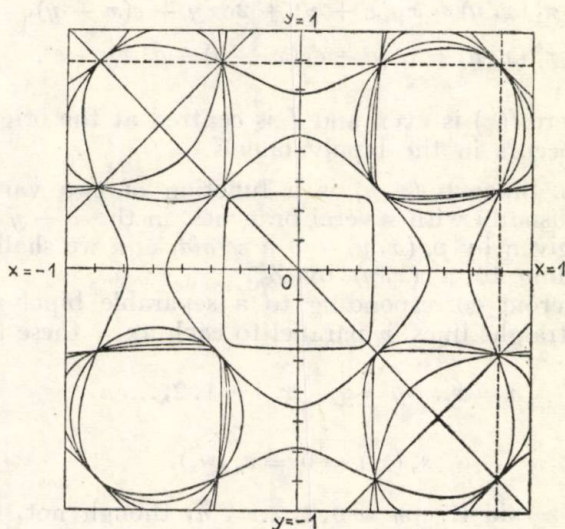


Fig. 1

The zeroids  $Z_m^m$ ,  $m = 0, 1, 2$  thus consist of a circle, the straight line  $x + y = 0$  and the pair of hyperbolas  $x^2y^2 = 1/9$ . For  $n = 3$ , the zeroids break up into the axes, the line  $x + y = 0$ , a circle, a pair of hyperbolas and an ellipse. For  $n = 4$ , however, the curves are much more complicated. The complete set of curves for  $n = 4$  is shown in Fig. 1 below. This exhibits clearly the properties listed earlier. The zeroid of  $S_4(x, y)$  consists of eight straight lines, two of which are shown dotted in the figure. The zeroid  $Z_4^0$  has equation

$$x^4y^4 - \frac{36}{49}x^2y^2 + \frac{18}{245}(x^2 + y^2) - \frac{9}{1225} = 0.$$

One part of this zeroid is the innermost „square“ curve cutting  $y = \pm x$  at points other than those of the set  $(x_r, \pm x_r)$ ,  $r = 1, 2, 3, 4$ .



$$(b) w(x, y) = 1, R = [0, 1] \times [0, 1].$$

The polynomials  $s_n(x)$  are obtained from those of (a) by a linear change of variable. We find

$$s_0(x) = 1, \quad s_1(x) = x - \frac{1}{2}, \quad s_2(x) = x^2 - x + \frac{1}{6}.$$

Hence

$$\pi_1^1 = x + y - 1, \quad \pi_1^0 = xy - \frac{1}{4}, \quad \pi_2^2 = (x + y)^2 - 2xy - (x + y) + \frac{1}{3},$$

$$\pi_2^1 = xy(x + y) - 2xy + \frac{1}{6}(x + y), \quad \pi_2^0 = x^2 y^2 - xy + \frac{1}{6}(x + y) - \frac{1}{36}.$$

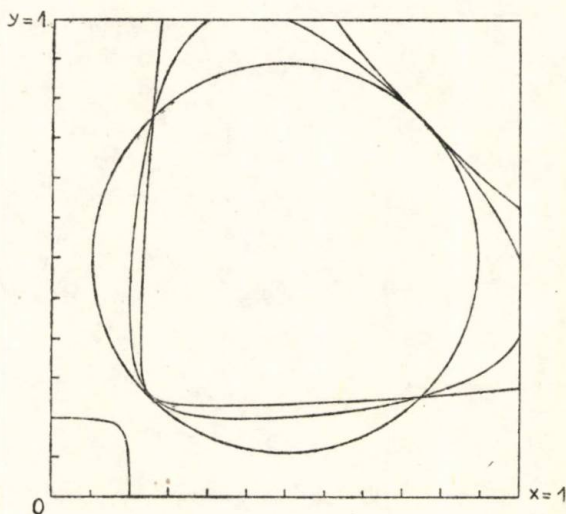


Fig. 2

The zeroids  $Z_2^m$ ,  $m = 0, 1, 2$  are shown in Fig. 2. It should be noted that  $Z_2^1$  has a branch outside  $R$ , but passing through the origin.

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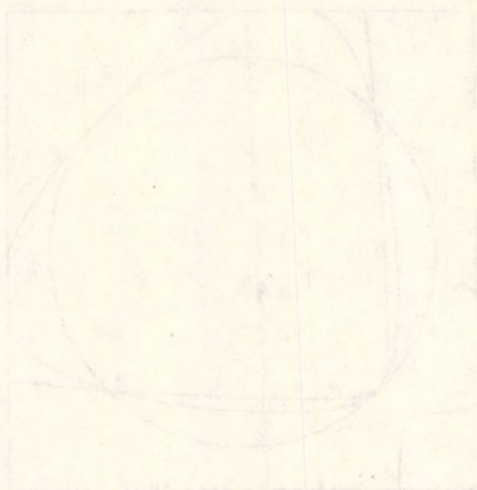
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## NOTĂ ASUPRA UNEI CLASE DE BIPOLINOAME

(Rezumat)

Se precizează proprietățile generale ale bipolinoamelor ortogonale introduse în [1] și se studiază proprietățile specifice unei clase de bipolinoame care prezintă un singur bipolinom separabil pentru fiecare grad.





## ON ODD DIMENSIONAL SUBMANIFOLDS OF A COMPLEX MANIFOLD

BY

AUREL BEJANCU

**0. Introduction.** Two classes of submanifolds immersed in a complex manifold have been intensively studied: complex submanifolds [7] and totally real submanifolds [5]. Recently we have initiated a study of the geometry of  $CR$  submanifolds in an almost Hermitian manifold [1], [2]. Thus, complex submanifolds and totally real submanifolds appear as particular cases of  $CR$  submanifolds.

Each real hypersurface of an almost Hermitian manifold is an example of  $CR$  submanifold which is neither complex submanifold nor totally real submanifold. Also, it is known that on a real hypersurface in an almost complex manifold there exists an almost contact structure [9]. Starting from this property, there have been obtained fundamental results on the geometry of a real hypersurface in a complex manifold, [6], [8], [10].

The purpose of the present paper is to initiate a study of the geometry of a class of odd dimensional submanifolds of an almost complex manifold. First, we prove the existence of a natural almost contact structure on each submanifold from this class. Then, by using the splitting of the tangent bundle of a such submanifold we get necessary and sufficient conditions for the natural almost contact structure in order to be normal.

Real hypersurfaces and some  $CR$  submanifolds of an almost Hermitian manifold belong to the class of submanifolds introduced in this paper. For real hypersurfaces such results have been obtained by Y. Tashiro [9], K. Yano and S. Ishihara [10].

**1.  $D$ -almost Contact Submanifolds of an Almost Complex Manifold.** Let  $N$  be an almost complex manifold of complex dimension  $n$  and  $M$  be a Riemannian submanifold of  $N$  of real dimension  $2m + 1$ . Denote by  $J$  the almost complex structure on  $N$  and by  $TM$  the tangent bundle to  $M$ . All manifolds and all mappings in the paper are considered to be differentiable of class  $C^\infty$ .

It is known, that  $M$  is said to be an almost contact manifold if it admits a tensor field  $f$  of type (1, 1), a vector field  $U$  and 1-form  $u$  on  $M$  satisfying [3]

$$(1.1) \quad f^2 = -I + u \otimes U$$

and

$$(1.2) \quad u(U) = 1,$$

where  $I$  denotes the identity morphism on  $TM$ . In what follows we need the tensor field  $N^{(1)}$  on  $M$

$$(1.3) \quad N^{(1)}(X, Y) = N_f(X, Y) + [(\nabla_X u)(Y) - (\nabla_Y u)(X)] U,$$

where  $N_f$  is the Nijenhuis tensor constructed from  $f$  by

$$(1.4) \quad N_f(X, Y) = [fX, fY] - f[fX, Y] - f[X, fY] + f^2[X, Y].$$

The almost contact structure is said to be normal if the tensor field  $N^{(1)}$  vanishes.

Now, suppose on  $M$  there exists a distribution  $D$  of real dimension  $2m$  which is invariant by  $J$ , that is,  $J(D_x) = D_x$  for each  $x \in M$ . The distribution  $D$  will be called the holomorphic distribution and  $M$  is said to be a  $D$ -almost complex submanifold of  $N$ .

About this class of submanifolds of odd dimension immersed in  $N$ , we can prove

**Theorem 1.1.** *Each  $D$ -almost complex submanifold  $M$  of an almost complex manifold  $N$  carries a naturally induced almost contact structure.*

**Proof.** There exists a vector field  $U$  on  $M$  such that  $JU$  is not tangent to  $M$ . In fact, otherwise,  $M$  admits an almost complex structure, which is never possible because  $M$  is of odd dimension. Hence  $U$  does not belong to the holomorphic distribution. This implies  $T_x M = D_x \oplus D_x^\perp$ , where  $D_x^\perp$  is the subspace generated by  $U_x$  at the point  $x \in M$ . Thus we have a supplementary distribution to  $D$  on  $M$ , denoted by  $D^\perp$  and of dimension one.

The distributions  $D$  and  $D^\perp$  are given by the projectors  $P$  and  $Q$  respectively. Then, for each vector field  $X$  on  $TM$  we put

$$(1.5) \quad QX = u(X)U.$$

Thus  $X$  has the unique decomposition

$$(1.6) \quad X = PX + u(X)U.$$

Now, we define the tensor field  $f$  on  $M$  by

$$(1.7) \quad fX = JPX,$$

for each  $X \in TM$ . Applying  $J$  to both sides of (1.6) we get

$$(1.8) \quad JX = fX + u(X)JU.$$

Taking into account that  $J^2 = -I$ , from (1.8) we have

$$f^2X + u(fX)JU - u(X)U = -X,$$

which implies

$$(1.9) \quad f^2 = -I + u \otimes U; \quad u \circ f = 0.$$



Next, substituting  $X$  by  $U$  into (1.8), we obtain

$$(1.10) \quad u(U) = 1; \quad fU = 0.$$

The relations (1.9) and (1.10) prove that  $(f, U, u)$  is an almost contact structure on  $M$ . The proof is complete.

From now on, the submanifold  $M$  will be called a  $D$ -almost contact submanifold of  $N$  and  $(f, U, u)$  will be called the natural almost contact structure on  $M$ .

Now, we shall say something about the equivalence of two natural almost contact structures on  $M$ .

Two almost contact structures  $(f', U', u')$  and  $(f, U, u)$  on  $M$  are said to be equivalent, if they are related by

$$(1.11) \quad f' = f; \quad U' = kU; \quad u' = \frac{1}{k}u,$$

for some real function  $k \neq 0$  on  $M$ .

**Theorem 1.2.** Any two natural almost contact structures  $(f', U', u')$  and  $(f, U, u)$  defined on  $M$  are equivalent to each other.

**Proof.** First we notice that  $f$  and  $f'$  are defined by the formula (1.7) which depends only on fixed operators  $J$  and  $P$ . Thus, the first relation of (1.11) is satisfied. Now, since  $U$  and  $U'$  does not belong to  $D$  and both are non null vector fields on  $M$ , there exists a real function  $k \neq 0$  such that  $U' = kU$ . Finally, from (1.5) we have  $u'(X) U' = u(X) U$ , which implies the third relation of (1.11).

**Remarks.** 1. Each real hypersurface of an almost complex manifold is a  $D$ -almost contact manifold [9].

2. Each  $CR$  submanifold of odd dimension with  $\dim D^\perp = 1$  immersed in an almost Hermitian manifold is a  $D$ -almost contact manifold [1], [2], [4].

3. There exist almost contact submanifolds which are not  $D$ -almost contact submanifolds. We shall give here an example.

Let us take three copies  $S_1, S_2, S_3$  of the circle and consider each one immersed in the Euclidean plane  $E^2$ . Then,  $S_1 \times S_2 \times S_3$  has an almost contact structure, but it is totally real immersed in  $E^6$  [5].

**2.  $D$ -almost Contact Submanifolds of a Complex Manifold.** In the sequel, suppose  $N$  is a complex manifold. In this case, the Nijenhuis tensor  $N_J$  constructed from the almost complex structure  $J$  vanishes and there exists a symmetric affine connection  $\tilde{\nabla}$  such that  $\tilde{\nabla}J = 0$ .

Denote by  $\nu$  a vector bundle on  $M$  which is supplementary to the tangent bundle of  $M$  into the tangent bundle of  $N$  and contains the vector field  $V = JU$ . We choose a vector bundle  $\mu$  supplementary to that one generated by  $V$  into  $\nu$ .

Now, we take the local field of frames on  $\nu$

$$(2.1) \quad \{V, V_1, \dots, V_s, \bar{V}_1 = JV_1, \dots, \bar{V}_s = JV_s\},$$

where  $s = n - m - 1$  and  $V_a, \bar{V}_a \in \mu$  for  $a = 1, \dots, s$ . The affine connection induced by  $\tilde{\nabla}$  on  $M$  will be denoted by  $\nabla$ .

Then, the equation of Gauss can be written in the following convenient form

$$(2.2) \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)V + \sum_{a=1}^s \{h^a(X, Y)V_a + \bar{h}^a(X, Y)\bar{V}_a\},$$

where  $h, h^a$ , and  $\bar{h}^a$  are the second fundamental tensors of type  $(0, 2)$  of the  $D$ -almost contact submanifold with respect to the affine normals  $V, V_a$  and  $\bar{V}_a$  respectively.

Also, we shall make use of the following equation of Weingarten

$$(2.3) \quad \tilde{\nabla}_X V = -HX + t(X)V + \sum_{a=1}^s \{t^a(X)V_a + \bar{t}^a(X)\bar{V}_a\},$$

where  $H$  is a tensor field of type  $(1, 1)$  on  $M$ , called the fundamental tensor of Weingarten with respect to  $V$ , and  $t, t^a, \bar{t}^a$  are 1-forms on  $M$ . Of course, there exist some other equations of Weingarten corresponding to other normals, but (2.3) is sufficient for our purpose.

Since  $\tilde{\nabla}$  is without torsion, then the affine connection  $\nabla$  has no torsion and the fundamental tensors  $h, h^a$  and  $\bar{h}^a$  are all symmetric.

Now, applying the operator  $\tilde{\nabla}_Y$  to (1.8) and by using (2.2) and (2.3), we obtain

$$(2.4) \quad (\nabla_Y f)(X) = u(X)HY - h(X, Y)U,$$

$$(2.5) \quad (\nabla_Y u)(X) = -h(Y, fX) - u(X)t(Y),$$

$$(2.6) \quad h^a(X, Y) = \bar{h}^a(Y, fX) + u(X)\bar{t}^a(Y),$$

$$(2.7) \quad \bar{h}^a(X, Y) = -h^a(Y, fX) - u(X)t^a(Y),$$

for all vector fields  $X, Y$  tangent to  $M$ .

In the same way, applying  $\tilde{\nabla}_Y$  to the relation  $JV = -U$ , we get

$$(2.8) \quad (\nabla_Y U) = fHY + t(Y)U,$$

$$(2.9) \quad h(Y, U) = u(HY),$$

$$(2.10) \quad t^a(Y) = -\bar{h}^a(Y, U),$$

$$(2.11) \quad \bar{t}^a(Y) = h^a(Y, U).$$

The remaining part of this paragraph is devoted to getting necessary and sufficient conditions for the natural almost contact structure  $(f, U, u)$  in order to be normal. For this we need some notations and definitions.

Let  $v$  be a 1-form on  $M$  and  $T$  be a tensor field on  $M$  of type  $(1, 1)$ . Then, the exterior product of  $v$  and  $T$  is a tensor field of type  $(1, 2)$  denoted by  $v \wedge T$  and defined by



$$(2.12) \quad (v \wedge T)(X, Y) = v(X)T(Y) - v(Y)T(X),$$

for all  $X, Y \in TM$ . Also, denote by  $A$  the tensor field of type  $(1, 1)$  defined by

$$(2.13) \quad AX = t(X)U, \text{ for all } X \in TM.$$

Now we can state

**Theorem 2.1.** *Let  $M$  be a  $D$ -almost contact submanifold of the complex manifold  $N$ . Then, the following three conditions are equivalent to each other.*

- (a) *The natural almost contact structure  $(f, U, u)$  on  $M$  is normal;*
- (b)  *$u \wedge (A - Hf + fH) = 0$ ;*
- (c) *The following two relations are satisfied on  $M$ :*

$$(2.14) \quad u \wedge (fH - PHf) = 0,$$

$$(2.15) \quad u \wedge (A - QHf) = 0.$$

**Proof.** Since  $\nabla$  is an affine connection without torsion, from (1.3) we obtain

$$(2.16) \quad N^{(1)}(X, Y) = (\nabla_{fX}f)Y - (\nabla_{fY}f)X - f(\nabla_X f)Y + f(\nabla_Y f)X + [(\nabla_X u)Y - (\nabla_Y u)X]U.$$

Next, by using (2.4) and (2.5) in (2.16) we have

$$(2.17) \quad N^{(1)} = u \wedge (A - Hf + fH),$$

which proves the equivalence of (a) and (b). Finally (2.14) and (2.15) are equivalent to the condition (b) because they are obtained from (b) by applying the projectors  $P$  and  $Q$  respectively. This completes the proof.

Now, by using the splitting of the tangent bundle  $TM = D \oplus D^\perp$ , we can prove

**Theorem 2.2.** *The natural almost contact structure  $(f, U, u)$  on  $M$  is normal if and only if we have*

$$(2.18) \quad N^{(1)}(X, U) = 0,$$

for each vector field  $X \in D$ .

**Proof.** First we show

$$(2.19) \quad N^{(1)}(X, Y) = 0, \text{ for all } X, Y \in D.$$

For this we notice that on the holomorphic distribution we have.  $f = J$ . Hence  $N_f(X, Y) = N_J(X, Y) = 0$ , because the almost complex structure  $J$  is integrable on  $N$ . Also, we see that (2.4) and (2.5) become

$$(2.20) \quad (\nabla_Y f)(X) = -h(X, Y)U,$$

$$(2.21) \quad (\nabla_Y u)(X) = -h(Y, fX),$$

for all  $X, Y \in D$ . Then, (2.19) follows from (2.16) by using (2.20) and (2.21)

Finally,  $N^{(1)}(U, U) = 0$  since  $N^{(1)}$  is an anti-symmetric tensor field. Therefore, the theorem follows from the decomposition (1.6).

There exists another important tensor field which appears in the study of normal almost contact structures. It is a tensor field of type (1.1) denoted by  $N^{(3)}$  and defined by

$$(2.22) \quad N^{(3)}(X) = (L_U f)(X),$$

where  $L_U$  is the Lie differentiation with respect to  $U$ .

Now, we can state

**Theorem 2.3.** *The natural almost contact structure  $(f, U, u)$  on  $M$  is normal if and only if  $N^{(3)} = 0$  on the holomorphic distribution.*

**Proof.** It is well known that  $N^{(1)} = 0$  implies  $N^{(3)} = 0$  ([3], p. 50). Conversely, if  $N^{(3)} = 0$  on the holomorphic distribution, as we have seen in the previous theorem, we have to prove  $N^1(X, U) = 0$  for each  $X \in D$ .

First, by using (1.9), from (1.3) we get

$$N^{(1)}(X, U) = [U, X] + f[U, fX] - [U(u(X))]U.$$

But  $u(X) = 0$  for each  $X \in D$ . Hence the above equation becomes

$$(2.23) \quad N^{(1)}(X, U) = [U, X] + f[U, fX].$$

Next, by using  $N^{(3)} = 0$  and taking into account the definition of Lie differentiation operator we obtain

$$(2.24) \quad [U, fX] - f[U, X] = 0,$$

which transforms (2.23) into

$$(2.25) \quad N^{(1)}(X, U) = [U, X] + f^2[U, X] = u([U, X])U.$$

The last relation from (2.25) follows from (1.9).

Finally, applying  $u$  to (2.24) we get  $u([U, fX]) = 0$  for each  $X \in D$ . Since  $f = J$  on  $D$  and  $J$  is an automorphism on  $D$  we have  $u([U, X]) = 0$ . Therefore, by (2.25) we get  $N^{(1)}(X, U) = 0$ . The proof is complete.

**Theorem 2.4.** *Let  $M$  be a  $D$ -almost contact submanifold of a complex manifold  $N$ . Then, the following three conditions are equivalent to each other.*

- (a) *The natural almost contact structure  $(f, U, u)$  is normal;*
- (b)  *$Hf - fH = A$  on the holomorphic distribution;*
- (c)  *$t = uHf$  and  $fH = PHf$  on the holomorphic distribution.*

**Proof.** By the Theorem 2.2. and by (2.14) we see that the almost contact structure  $(f, U, u)$  is normal if and only if

$$(2.26) \quad [u \wedge (A - Hf + fH)](X, U) = 0,$$

for each  $X \in D$ . Since  $u(X) = 0$  and  $u(U) = 1$ , (2.26) is equivalent to the condition (b) of the theorem. The equivalence of the conditions (b) and (c) follows by means of the projectors  $P$  and  $Q$ . The proof is complete.

The vector field  $U$  is said to be parallel along the holomorphic distribution if we have  $\nabla_X U = 0$  for each  $X \in D$ .

Now, we can prove



**Theorem 2.5.** *If  $U$  is parallel along the holomorphic distribution then the natural almost contact structure  $(f, U, u)$  is normal if and only if the fundamental tensor of Weingarten  $H$  vanishes on the holomorphic distribution.*

**Proof.** From (2.8) we get  $t = 0$  and  $fH = 0$  on the holomorphic distribution. Thus, by the condition (b) of the Theorem 2.4.,  $(f, U, u)$  is normal if and only if  $Hf = 0$  on  $D$ . The last equality is equivalent to  $H = 0$  because  $f = J$  on  $D$ . This completes the proof.

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#### ASUPRA SUBVARIETĂȚILOR DE DIMENSIUNE PARĂ ALE UNEI VARIETĂȚI COMPLEXE

(Rezumat)

Se studiază geometria unei clase de subvarietăți de dimensiune pară a unei varietăți aproape complexe.

Se dovedește existența unei structuri naturale de aproape contact pe fiecare subvarietate a acestei clase iar apoi se stabilesc condițiile necesare și suficiente pentru ca o structură normală de aproape contact să fie normală.

The first of these is the fact that the distribution of the various species of the genus is not uniform. It is found in all parts of the world, but is particularly abundant in the tropics. The second is the fact that the distribution of the various species of the genus is not uniform. It is found in all parts of the world, but is particularly abundant in the tropics. The third is the fact that the distribution of the various species of the genus is not uniform. It is found in all parts of the world, but is particularly abundant in the tropics.

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## ON THE EXISTENCE OF A KERNEL IN A STRONG CONNECTED DIGRAPH

BY

DĂNUȚ MARCU

1. In this paper we show that a strong connected digraph  $D = \langle \mathcal{N}, \mathcal{A} \rangle$  contains a kernel, if it exists a unique partition  $\mathcal{P} = \{\mathcal{N}_1, \mathcal{N}_2\}$  of  $\mathcal{N}$ , so that, every arc of  $\mathcal{A}$  links a node of  $\mathcal{N}_1$  with a node of  $\mathcal{N}_2$ .

2. Let  $D = \langle \mathcal{N}, \mathcal{A} \rangle$  be an oriented finite graph (digraph [1]) with  $\mathcal{N} = \{n_1, n_2, \dots, n_p\}$  the set of nodes and  $\mathcal{A} = \{a_1, a_2, \dots, a_q\}$  the set of arcs. Denoting by  $\mathcal{C}[D]$  the set of elementary circuits of  $D$ , and by  $C[D]$  the set of elementary cycles, we attach, arbitrary, to every arc  $a_\alpha$ ,  $\alpha = 1, 2, \dots, q$ , a number  $\varepsilon_\alpha = 1$  or  $-1$ .

For an elementary cycle ( $\sigma$ ), circuit ( $\omega$ ), path ( $\gamma$ ) or chain ( $L$ ), we denote by  $\mathcal{A}(\sigma)$ ,  $\mathcal{A}(\omega)$ ,  $\mathcal{A}(\gamma)$ ,  $\mathcal{A}(L)$  their set of arcs.

Lemma 1. If  $\omega$  is a circuit with  $\prod_{a_\alpha \in \mathcal{A}(\omega)} \varepsilon_\alpha < 0$ , then it exists an elementary circuit  $\tilde{\omega}(\mathcal{A}(\tilde{\omega}) \subseteq \mathcal{A}(\omega))$ , so that  $\prod_{a_\beta \in \mathcal{A}(\tilde{\omega})} \varepsilon_\beta < 0$ .

Proof. Evidently,  $\omega$  contains an elementary circuit  $\omega^*$ . If  $\prod_{a_\beta \in \mathcal{A}(\omega^*)} \varepsilon_\beta < 0$ , the lemma is proved, else, we consider the circuit  $\omega^{**}$  for which we have  $\mathcal{A}(\omega^{**}) = \mathcal{A}(\omega) \setminus \mathcal{A}(\omega^*)$  and  $\prod_{a_\alpha \in \mathcal{A}(\omega^{**})} \varepsilon_\alpha < 0$ .

Repeating this process (putting  $\omega := \omega^{**}$ ) we obtain finally an elementary circuit  $\tilde{\omega}$  with  $\prod_{a_\beta \in \mathcal{A}(\tilde{\omega})} \varepsilon_\beta < 0$ . (Q.E.D.)

Lemma 2. (F. Harary [2]). Let  $D = \langle \mathcal{N}, \mathcal{A} \rangle$  be a digraph so that  $\prod_{a_\alpha \in \mathcal{A}(\sigma)} \varepsilon_\alpha > 0$  for all  $\sigma \in C[D]$ . If  $n$  and  $m$  are two distinct nodes of  $\mathcal{N}$ ,

then, for every two distinct elementary chains  $L_1, L_2$  between  $n$  and  $m$ , we have

$$\prod_{a_\alpha \in \mathcal{A}(L_1)} \varepsilon_\alpha \prod_{a_\beta \in \mathcal{A}(L_2)} \varepsilon_\beta > 0.$$

Let  $C_1, C_2, C_3, \dots, C_M$  be the strong connected components of  $D$ .



**Lemma 3.** *The product  $\prod_{\alpha \in \mathcal{A}(\omega)} \varepsilon_\alpha$  is positive for all  $\omega \in \mathcal{O}[D]$ , if and only if the product  $\prod_{\beta \in \mathcal{A}(\sigma)} \varepsilon_\beta$  is positive for all  $\sigma \in C[\mathbf{C}_i]$  and  $i = 1, 2, \dots, M$ .*

**Proof.** Suppose that exist  $i_0 \in \{1, 2, \dots, M\}$  and  $\sigma_0 \in C[\mathbf{C}_{i_0}]$  so that  $\prod_{\beta \in \mathcal{A}(\sigma_0)} \varepsilon_\beta < 0$ , and consider  $\sigma_0$  as the form  $\sigma_0 = [a_{t_1}, a_{t_2}, \dots, a_{t_N}]$

Choosing an arbitrary sense for the orientation of the arcs, we attach to every  $a_{t_k}, k = 1, 2, \dots, N$  the label  $m_{t_k}$  so that:

$$m_{t_k} = \begin{cases} + & \text{if the arc } a_{t_k} \text{ is oriented in the chosen sense,} \\ - & \text{otherwise.} \end{cases}$$

Let  $a_{t_{k_0}}$  be an arbitrary arc  $A(\sigma_0)$ . Because  $\mathbf{C}_{i_0}$  is a strong connected component of  $D$ , there exists an elementary path  $\gamma_{k_0} [\Delta^-(a_{t_{k_0}}), \Delta^+(a_{t_{k_0}})]$  (if an arc  $a$  has the form  $a = \langle n, m \rangle$ , we denote  $\Delta^+(a) = n, \Delta^-(a) = m$ ).

Evidently,  $(\gamma_{k_0} \cup a_{t_{k_0}}) \in \mathcal{O}[D]$ , and, from hypothesis, we have

$$(1) \quad \prod_{\alpha \in \mathcal{A}(\gamma_{k_0})} \varepsilon_\alpha \varepsilon_{t_{k_0}} > 0.$$

We construct the circuit  $\omega_0$ , replacing in  $\sigma_0$  every arc  $a_{t_k}$  for which  $m_{t_k} = -$ , by the path  $\gamma_k$ . From (1) we obtain  $\prod_{\alpha \in \mathcal{A}(\omega_0)} \varepsilon_\alpha < 0$ .

But, having in view Lemma 1,  $\omega_0$  contains an elementary circuit  $\tilde{\omega}_0 (\mathcal{A}(\tilde{\omega}_0) \subseteq \mathcal{A}(\omega_0))$  for which  $\prod_{\alpha \in \mathcal{A}(\tilde{\omega}_0)} \varepsilon_\alpha < 0$ ; contradiction with the hypothesis. Hence,  $\prod_{\beta \in \mathcal{A}(\sigma)} \varepsilon_\beta$  is positive for all  $\sigma \in C[\mathbf{C}_i]$  and  $i = 1, 2, \dots, M$ . (Q.E.D.)

Reciprocally, because every elementary circuit is contained in a strong connected component, we have  $\prod_{\alpha \in \mathcal{A}(\omega)} \varepsilon_\alpha > 0$ , for all  $\omega \in \mathcal{O}[D]$ . (Q.E.D.)

**Corollary 1.** *Let  $D = \langle \mathcal{N}, \mathcal{A} \rangle$  be a strong connected digraph. We have  $\prod_{\alpha \in \mathcal{A}(\omega)} \varepsilon_\alpha > 0$  for all  $\omega \in \mathcal{O}[D]$  if and only if  $\prod_{\beta \in \mathcal{A}(\sigma)} \varepsilon_\beta > 0$ , for all  $\sigma \in C[D]$ .*

**Proof.** If  $D$  is strong connected we have  $M = 1$ , and having in view Lemma 3, the Corollary is proved.

**3. Theorem 1.** *In a strong connected graph every elementary circuit contains an even number of arcs if and only if every elementary cycle contains an even number of arcs.*

**Proof.** The theorem is proved putting  $\varepsilon_i = -1, i = 1, 2, \dots, q$ , and having in view the Corollary 1.

**Theorem 2.** (D. König [3]). *Every elementary cycle of a digraph contains an even number of arcs if and only if it exists a partition  $\mathcal{P} = \{\mathcal{N}_1, \mathcal{N}_2\}$  of  $\mathcal{N}$ , so that every arc of  $\mathcal{A}$  links a node of  $\mathcal{N}_1$  with a node of  $\mathcal{N}_2$ .*

**Theorem 3.** *If  $D = \langle \mathcal{N}, \mathcal{A} \rangle$  is a strong connected digraph, then, every elementary cycle contains an even number of arcs if and only if*



it exists a unique partition  $\mathcal{P} = \{\mathcal{N}_1, \mathcal{N}_2\}$  of  $\mathcal{N}$  so that, every arc of  $\mathcal{A}$  links a node of  $\mathcal{N}_1$  with a node of  $\mathcal{N}_2$ .

**Proof.** Suppose that the partition  $\mathcal{P}$  is not unique. In this case it exists another partition  $\mathcal{P}' = \{\mathcal{N}'_1, \mathcal{N}'_2\}$ ,  $\mathcal{P}' \neq \mathcal{P}$ , so that, every arc of  $\mathcal{A}$  links a node of  $\mathcal{N}'_1$  with a node of  $\mathcal{N}'_2$ . If  $\mathcal{N}'_1 \cap \mathcal{N}_2 = \emptyset$ , evidently  $\mathcal{P}$  is unique, hence,  $\mathcal{N}'_1 \cap \mathcal{N}_1 \neq \emptyset$ . Let  $n_0^{(1)} \in \mathcal{N}_1$  so that  $n_0^{(1)} \notin \mathcal{N}'_1$  (in this case  $n_0^{(1)} \in \mathcal{N}'_2$ ) and  $m_0^{(1,2)} \in \mathcal{N}_1 \cap \mathcal{N}'_1$ .

Related to the partition  $\mathcal{P}$ , every elementary chain  $L_1$  (it exists because  $D$  is strong connected) between  $n_0^{(1)}$  and  $m_0^{(1,2)}$  contains an even number of arcs, and related to the partition  $\mathcal{P}'$ , every elementary chain  $L_2$  between  $n_0^{(1)}$  and  $m_0^{(1,2)}$  contains an odd number of arcs.

Hence, we have  $\prod_{\alpha \in \mathcal{A}(L_1)} \varepsilon_\alpha \prod_{\beta \in \mathcal{A}(L_2)} \varepsilon_\beta < 0$ ; contradiction with Lemma 2,

and the theorem is proved, having in view Theorem 2.

**Theorem 4.** If  $D = \langle \mathcal{N}, \mathcal{A} \rangle$  is a strong connected digraph, then every elementary circuit of  $D$  contains an even number of arcs if and only if it exists a unique partition  $\mathcal{P} = \{\mathcal{N}_1, \mathcal{N}_2\}$  of  $\mathcal{N}$ , so that, every arc of  $\mathcal{A}$  links a node of  $\mathcal{N}_1$  with a node of  $\mathcal{N}_2$ .

**Proof.** The proof is evidently, having in view Theorem 1 and Theorem 3.

**Theorem 5** (M. Richardson [5]). A digraph without odd elementary circuits contains a kernel.

**Theorem 6.** Let  $D = \langle \mathcal{N}, \mathcal{A} \rangle$  be a strong connected digraph. If it exists a unique partition  $\mathcal{P} = \{\mathcal{N}_1, \mathcal{N}_2\}$  of  $\mathcal{N}$ , so that every arc of  $\mathcal{A}$  links a node of  $\mathcal{N}_1$  with a node of  $\mathcal{N}_2$ , then,  $D$  contains a kernel.

**Proof.** The proof is evidently from Theorem 4 and Theorem 5.

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## ASUPRA EXISTENȚEI UNUI NUCLEU ÎNTR-UN DIGRAF TARE CONEX

(Rezumat)

Se arată că un digraf tare conex  $D = \langle \mathcal{N}, \mathcal{A} \rangle$  conține un nucleu dacă există o partiție unică  $\mathcal{P} = \{\mathcal{N}_1, \mathcal{N}_2\}$  a mulțimii  $\mathcal{N}$ , astfel încât fiecare arc al lui  $\mathcal{A}$  leagă un nod din  $\mathcal{N}_1$  cu un nod din  $\mathcal{N}_2$ .





## SUR UN OPÉRATEUR MATRICIEL

PAR

DAVID RIMER

Soit l'ensemble

$$(1) \quad X_n^r = Q^r \times R^{n-r}, \quad (0 < r \leq n),$$

où  $Q$  et  $R$  sont les ensembles des nombres rationnels, et respectivement, des réels. Considérons l'application

$$(2) \quad \varphi : X_n^r \rightarrow R^m \quad (1 \leq m < n)$$

donnée par

$$(2') \quad Y = MX, \quad (X \in X_n^r; \quad Y \in R^m),$$

où  $M$  est une matrice de type  $m \times n$  sur  $R$ .

En notant

$$(3) \quad Y = \mathcal{I}_m \varphi,$$

nous nous proposons à déterminer, s'il est possible :

A) la matrice  $M$  et B) deux topologies, sur  $X_n^r$  respectivement sur  $Y$ , tellement que l'application

$$(4) \quad \varphi_1 : X_n^r \rightarrow Y$$

soit un homéomorphisme.

*Le problème présente un certain intérêt théorique et pratique, vu qu'on ne peut pas réaliser un homéomorphisme,  $R^n \rightarrow R^m$  ( $m \neq n$ ).*

A) L'objectif proposé implique que  $\text{Ker } \varphi = 0$ , et donc, le système

$$(5) \quad MX = 0, \quad (X \in X_n^r)$$

ait la solution unique

$$(6) \quad X = 0.$$

On distingue deux cas :  $m > 1$  et  $m = 1$ .

**Théorème 1.** *Une condition suffisante afin que l'application (2), avec  $m > 1$ , donnée par (2'), soit injective, est que la matrice  $M$  ait la forme*

$$(7) M = \begin{bmatrix} a_1^1 & \dots & a_{j_1}^1 & \dots & 0 & \dots & 0 \\ b_1^2 & \dots & b_{j_1}^2 & a_{j_1+1}^2 & \dots & a_{j_2}^2 & \dots & 0 & \dots & 0 \\ b_1^3 & \dots & \dots & b_{j_2}^3 & a_{j_2+1}^3 & \dots & a_{j_3}^3 & \dots & 0 & \dots & 0 \\ b_1^i & \dots & \dots & \dots & b_{j_{i-1}}^i & a_{j_{i-1}+1}^i & \dots & a_{j_i}^i & \dots & 0 & \dots & 0 \\ c_1^{i+1} & \dots & \dots & \dots & \dots & \dots & \dots & c_{j_i}^{i+1} & d_{j_i+1}^{i+1} & \dots & d_n^{i+1} \\ c_1^m & \dots & \dots & \dots & \dots & \dots & \dots & c_{j_i}^m & d_{j_i+1}^m & \dots & d_n^m \end{bmatrix}$$

où l'on a

$\alpha)$   $1 \leq i \leq m$ ;  $2 \leq j_i \leq r$ ;  $0 \leq n - j_i \leq m - i$ ; si  $i = 1$ , ou si  $i = m$ , les éléments  $b$ , respectivement  $c$  et  $d$ , manquent totalement;

$\beta)$  dans chacune des lignes  $l$  ( $l = 1, 2, \dots, i$ ), les éléments  $a_p^l$  ( $p = j_{l-1} + 1, j_l$ , où  $j_0 = 0$ ) forment une famille libre sur  $Q$ ;

$\gamma)$   $\text{rang}(d_k^h) = n - j_i$ , ( $h = \overline{i+1, m}$ );  $k = \overline{j_i+1, n}$ .

Démonstration. I) Avec (7), la première équation du système (5) est  $\sum_{p=1}^{j_1} a_p^1 x^p = 0$ . Puisque  $x^p \in Q$  ( $p \leq j_1 \leq r$ ) et vu  $\beta)$ , cette équation à la solution unique  $x^1 = \dots = x^{j_1} = 0$ . Si  $i = 1$ , on passe à III). Si  $i > 1$ , la deuxième équation du système (5) est  $\sum_{p=j_1+1}^{j_2} a_p^2 x^p = 0$  (car  $x^1 = \dots = x^{j_1} = 0$ ). Vu que  $x^p \in Q$  et  $\beta)$ , on obtient,  $x^{j_1+1} = \dots = x^{j_2} = 0$ . Si  $i = 2$ , on passe à II). Si  $i > 2$ , on continue de la même manière et, après  $i$  étapes, on a

$$(8) \quad x^1 = x^2 = \dots = x^{j_i} = 0.$$

(II). Si  $i = m$ , de  $\alpha)$  il résulte  $m - i = 0$ ,  $n = j_i = r$  et (8) est justement (6).

III). Si  $i < m$ , alors  $n - j_i > 0$  et les dernières  $m - i$  équations de (5) constituent le système homogène  $\sum_{k=j_i+1}^n d_k^h x^k = 0$  ( $h = \overline{i+1, m}$ ). Vu  $\gamma)$ , ce système a la solution unique

$$(9) \quad x^{j_i+1} = \dots = x^n = 0.$$

Les égalités (8) et (9) constituent justement (6). Le théorème est démontré.

Corrolaire 1. Des inégalités  $\alpha)$  il en résulte  $0 \leq n - r < m$ ;  $n \geq 3$ .

Corrolaire 2. Pour  $r = 2$ , on a  $m = n - 1$ .

Corrolaire 3. L'application (4), avec la matrice  $M$  donnée dans le Théorème 1 et  $m > 1$ , est bijective.

Notons par

$$(10) \quad \varphi_1^{-1}: \bar{Y} \rightarrow X_n^r$$

l'inverse de l'application (4).

Étant donné  $Y_0 \in \bar{Y}$  et la matrice  $M$ , on peut déterminer effectivement  $X_0 = \varphi_1^{-1}(Y_0)$  à l'aide du



*Algorithme 1.* I) On considère la première égalité de (2'),  $\sum_{p=1}^{j_1} a_p^1 x^p = y_0^1$ . Vu que  $y_0^1$  est un élément de l'espace linéaire, à  $j_1$  dimensions, sur  $\mathcal{Q}$ , espace ayant pour base  $\{a_1^1, \dots, a_{j_1}^1\}$ , on a directement  $x_0^1, \dots, x_0^{j_1}$ . Si  $i = 1$ , on passe à III). Si  $i > 1$ , on considère la deuxième égalité de (2')

$\sum_{p=j_1+1}^{j_2} a_p^2 x^p = y_0^2 - \sum_{p=1}^{j_1} b_p^1 x_0^p$ . Le second membre est un élément de l'espace linéaire à  $j_2 - j_1$  dimensions, sur  $\mathcal{Q}$ , espace ayant pour base  $\{a_{j_1+1}^2, \dots, a_{j_2}^2\}$ . Alors on a directement  $x_0^{j_1+1}, \dots, x_0^{j_2}$ .

De cette manière on obtient successivement, en  $i$  étapes,  $x_0^1, x_0^2, \dots, x_0^{j_i}$ .

II). Si  $i = m$ , alors  $j_i = r$ ,  $n = r$  et l'algorithme est fini.

III). Si  $i < m$ , les dernières  $m - i$  égalités de (2') forment un système à  $m - i$  équations et  $n - j_i$  inconnues :

$$\sum_{k=j_i+1}^n d_k^h x^k = y_0^h - \sum_{s=1}^{j_i} c_s^h x_0^s \quad (h = \overline{i+1, m}).$$

Vu  $\gamma)$ , on obtient uniquement  $x_0^{j_i+1}, \dots, x_0^n$ .

*Exemple 1.* Considérons l'application  $\varphi: X_6^4 = \mathcal{Q}^4 \times R^2 \rightarrow R^4$  donnée par  $Y = MX$ , avec

$$M = \begin{bmatrix} -\sqrt{3} & 3 & 0 & 0 & 0 & 0 \\ 2 & 6 & \frac{1}{2} & \sqrt{5} & 0 & 0 \\ 0 & -1 & 2 & 1 & \sqrt{5} & \sqrt[3]{4} \\ -1 & -2 & 3 & 4 & -6 & 1 \end{bmatrix} \quad \text{et soit } Y_0 = \begin{bmatrix} 1 + \frac{\sqrt{3}}{2} \\ \frac{1}{2} + \sqrt{5} \\ \frac{5}{3} \\ \frac{5}{6} - 6\sqrt{5} - \sqrt[3]{2} \end{bmatrix}$$

Nous allons déterminer  $X_0 \in X_6^4$ , tel que  $\varphi(X_0) = Y_0$ .

Remarquons que toutes les conditions de l'hypothèse du Théorème 1 sont satisfaites.

La première équation du système  $MX = Y_0$  est  $-\sqrt{3}x^1 + 3x^2 = 1 + \frac{\sqrt{3}}{2}$ , ( $x^1, x^2 \in \mathcal{Q}$ ).

Elle est équivalente à  $\sqrt{3} \left( x^1 + \frac{1}{2} \right) + 3 \left( -x^2 + \frac{1}{3} \right) = 0$ , ( $x^1, x^2 \in \mathcal{Q}$ ) et a la solution unique

$x_0^1 = -\frac{1}{2}$ ,  $x_0^2 = \frac{1}{3}$ . Avec ces valeurs de  $x_0^1, x_0^2$ , la deuxième équation est  $\frac{1}{2}x^3 + \sqrt{5}x^4 = -\frac{1}{2} + \sqrt{5}$ , ( $x^3, x^4 \in \mathcal{Q}$ ); elle a la solution unique  $x_0^3 = -1$ ,  $x_0^4 = 1$ .

Les deux dernières équations forment le système

$$\sqrt{5}x^5 + \sqrt[3]{4}x^6 = 3, \quad -6x^5 + x^6 = -\sqrt[3]{2} - 6\sqrt{5}, \quad (x^5, x^6 \in R).$$

Il a la solution unique  $x_0^5 = \sqrt{5}$ ,  $x_0^6 = -\sqrt[3]{2}$ .

**Théorème 2.** Pour  $m = 1$ , l'application  $\varphi: X_n^r \rightarrow R$ , de la forme (2') est injective seulement si  $r < n$ , et les éléments de la matrice  $M$  sont linéairement indépendants sur  $Q$ .

**Démonstration.** La condition  $0 < r \leq n$ , implique  $n \geq 2$ . Pour  $m = 1$ , le système (5) se réduit à une seule équation  $\sum_{i=1}^n \alpha_i x^i = 0$ , ( $n \geq 2$ ;  $x^1, \dots, x^r \in Q$ ;  $x^{r+1}, \dots, x^n \in R$ ;  $\alpha_i \in R$ ). Si  $r < n$ , en aucun cas, cette équation ne peut pas avoir la solution unique  $x^1 = \dots = x^n = 0$ . Si  $r = n$ , l'injectivité est immédiate si  $\alpha_1, \dots, \alpha_n$  sont linéairement indépendants sur  $Q$ .

B) Une fois obtenue l'application bijective  $\varphi_1: X_n^r \rightarrow \bar{Y} \subset R^m$  ( $m > 1$ ), afin de réaliser le homéomorphisme cherché, nous allons organiser  $X_n^r$  et  $\bar{Y}$  en espaces topologiques. À ce but, nous définissons sur  $X_n^r$  la topologie  $\tau'$  induite par la topologie  $\tau$  de  $R^n$ ; les ouverts  $U$  de  $\tau$  étant les réunions d'intervalles  $n$ -dimensionnels ouverts, les ouverts  $U'$  de  $\tau'$  seront  $U' = U \cap X_n^r$ . Notons cet espace par  $(X_n^r, \tau')$ . Considérons sur  $\bar{Y}$  les ensembles  $V = \varphi(U')$ . On vérifie que les ensembles  $V$  définissent une topologie sur  $\bar{Y}$ . Notons par  $(Y, \tau'')$  l'espace topologique ainsi obtenu.

**Proposition 1.** Les applications  $\varphi_1: (X_n^r, \tau') \rightarrow (\bar{Y}, \tau'')$  et  $\varphi_1^{-1}: (\bar{Y}, \tau'') \rightarrow (X_n^r, \tau')$  sont continues sur  $X_n^r$ , respectivement sur  $\bar{Y}$ .

En effet, pour chaque ouvert  $V \subset Y$ ,  $\varphi_1^{-1}(V) = U'$  est un ouvert en  $(X_n^r, \tau')$  et pour chaque ouvert  $U' \subset X_n^r$ ,  $\varphi_1(U') = V$  est un ouvert en  $(\bar{Y}, \tau'')$ .

**Corrolaire 4.** L'application  $\varphi_1: (X_n^r, \tau') \rightarrow (\bar{Y}, \tau'')$  considérée là-dessus, est un homéomorphisme.

**Remarque 1.** Pour  $m = 2$  ou  $3$ , les applications  $\varphi_1$  et  $\varphi_1^{-1}$  peuvent être réalisées aussi à l'aide d'une construction graphique, respectivement par un modèle physique, qui est utile dans les sciences expérimentales [1].

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## ASUPRA UNUI OPERATOR MATRICEAL

(Rezumat)

Se realizează o aplicație injectivă  $\varphi: X_n^r = Q^r \times R^{n-r} \rightarrow R^m$  ( $0 < r \leq n$ ;  $1 \leq m < n$ ) prin intermediul unui operator matriceal. Notînd  $\bar{Y} = \mathcal{D}_m \varphi$ , se definește topologii pe  $X_n^r$  și pe  $\bar{Y}$  astfel ca  $\varphi_1: X_n^r \rightarrow \bar{Y}$  să fie un homeomorfism.



## SUR UNE DÉFINITION DES ESPACES GÉOMÉTRIQUES

PAR

DAN BORȘ

1. Dans son livre [1], le professeur V. C r u c e a n u donne une définition simple et intuitive de la notion d'espace affine associé à un espace vectoriel  $X$ , définition qui permet, par ses conséquences, de traiter la géométrie d'espaces affines et des variétés linéaires de ces espaces. En vue de la rapidité de l'exposition, l'auteur du livre [1] identifie facilement par isomorphisme l'espace vectoriel  $X$  avec l'ensemble  $V$  des vecteurs libres associés à l'espace affine.

Le but de cette Note est de relever le caractère géométrique de la définition des vecteurs libres et de construire une bijection  $V \rightarrow X$  qui transporte la structure vectorielle de  $X$  sur  $V$  et qui est un isomorphisme des espaces  $V$  et  $X$ .

Pour les besoins de la géométrie classique il est suffisant de considérer  $X = C^n$  (l'espace vectoriel complexe des  $n$ -uples ordonnées de nombres complexes).

2. **Définitions et conséquences immédiates.** Si  $\mathcal{A}$  est un ensemble nonvide de *points*, le couple ordonné  $(\mathcal{A}, C^n)$  est appelé *espace affine complexe* associé à l'espace vectoriel  $C^n$  s'il existe une fonction  $s : \mathcal{A}^2 \rightarrow C^n$  avec les propriétés :

$$A_1. \quad \forall (A, B, C) \in \mathcal{A}^3, \quad s(A, B) + s(B, C) = s(A, C);$$

$A_2. \quad \exists O \in \mathcal{A}$  ainsi que la restriction à  $\{O\} \times \mathcal{A}$  est une bijection ( $O$  s'appelle l'origine de la structure d'espace affine).

L'axiome  $A_2$  implique l'existence d'une bijection  $b : \mathcal{A} \rightarrow C^n$  donnée par  $b(M) = s(O, M)$  pour tout  $M \in \mathcal{A}$ . Les points  $I_k, 1 \leq k \leq n$ , ainsi que  $b(I_k) = e_k, 1 \leq k \leq n$ , s'appellent les *points unité* de l'espace affine  $\mathcal{A}$ ,  $(e_k)_{1 \leq k \leq n}$ , étant la base canonique de l'espace vectoriel  $C^n$ .

Les conséquences immédiates des axiomes  $A_1$  et  $A_2$  sont :

$$A_3. \quad \forall M \in \mathcal{A}, \quad s(M, M) = \theta_{C^n} \quad \text{et donc} \quad b(O) = \theta_{C^n}.$$

$$A_4. \quad \forall (A, B) \in \mathcal{A}^2, \quad s(B, A) = -s(A, B).$$

$$A_5. \quad \forall (A, B) \in \mathcal{A}^2, \quad s(A, B) = s(O, B) - s(O, A) = b(B) - b(A).$$

$$A_6. \quad \forall (A, B) \in \mathcal{A}^2, \quad s(A, B) = O \Rightarrow A = B.$$

$$A_7. \quad \forall x \in C^n, \quad \forall A \in \mathcal{A}, \quad \text{il existe un point unique } B \in \mathcal{A} \text{ tel que } x = s(A, B).$$



L'axiome  $A_7$  implique le fait que la fonction  $s$  est une surjection.

La relation d'équivalence  $R_s$  définie sur  $\mathcal{A}^2$  par la surjection  $s$ , c'est-à-dire  $(A, B) R_s (D, C) \Leftrightarrow s(A, B) = s(D, C)$  s'appelle la *relation d'équipollence* des couples ordonnés de points de l'espace affine  $\mathcal{A}$ , par rapport à la surjection  $s$ .

L'ensemble quotient  $V_n(s)$  de l'ensemble  $\mathcal{A}^2$  par rapport à la relation  $R_s$  s'appelle l'ensemble des vecteurs libres associés à l'espace affine complexe  $(\mathcal{A}, C^n)$  par rapport à la surjection  $s$ .

Un élément de  $V_n(s)$  s'appelle *vecteur libre* associé à l'espace affine  $(\mathcal{A}, C^n)$  par rapport à la surjection  $s$  et se note  $\overrightarrow{AB}$  s'il est la classe d'équivalence des couples ordonnés de points de  $\mathcal{A}$  équipollents modulo  $R_s$  à  $(A, B)$ .

**3. Proposition.** *Quelques soient les surjections  $s$  et  $s'$  de  $\mathcal{A}^2$  sur  $C^n$ , quelles satisfont les axiomes  $A_1$  et  $A_2$ , on ait  $V_n(s) = V_n(s')$ .*

En effet, soient  $O$  et  $O'$  les origines des deux structures d'espace affine définies par les surjections  $s$  et  $s'$  et  $I'_k$ ,  $1 \leq k \leq n$ , les points unité de l'espace affine d'origine  $O'$ .

Pour tout  $x = (x_k)_{1 \leq k \leq n}$ , de  $C^n$ , notons  $x_k = \text{pr}_k x$ ,  $1 \leq k \leq n$ , où  $\text{pr}_k$  est la projection  $\text{pr}_k x = (0, \dots, 0, x_k, 0, \dots, 0)$ , élément qui s'identifie à  $x_k$ .

Soit  $M$  un point quelconque de l'espace  $\mathcal{A}$ . Pour  $1 \leq k \leq n$ , les vecteurs  $\text{pr}_k s(O', M)$  et  $\text{pr}_k s'(O', M)$  appartiennent à l'espace vectoriel de dimension 1, généré par le vecteur  $e_k$  de la base canonique de  $C^n$ ; il existe alors  $a_k \in C$  tel que  $\text{pr}_k s'(O', M) = a_k \text{pr}_k s(O', M)$ ,  $1 \leq k \leq n$ . Nous avons

$$(1) \quad s(O', M) = (\text{pr}_k s(O', M))_{1 \leq k \leq n} = (a_k \text{pr}_k s'(O', M))_{1 \leq k \leq n},$$

et, en particulier,  $s(O', I'_k) = (0, \dots, 0, a_k, 0, \dots, 0)$ ,  $1 \leq k \leq n$ . La conséquence  $A_6$  et  $s'(O', I'_k) = e_k \neq \theta_{C^n}$ ,  $1 \leq k \leq n$ , impliquent  $O' \neq I'_k$ ,  $1 \leq k \leq n$ , d'où  $s(O', I'_k) \neq \theta_{C^n}$ ,  $1 \leq k \leq n$ , ou  $a_k \neq 0$ ,  $1 \leq k \leq n$ . En tenant compte de l'axiome  $A_1$ , l'égalité (1) s'écrit

$$(2) \quad s(O', M) = a_k \text{pr}_k s'(O', M)_{1 \leq k \leq n} = s(O, O').$$

En écrivant l'égalité (2) pour les points quelconques fixés  $A, B, C, D$ , on obtient  $s(A, B) - s(D, C) = (a_k \text{pr}_k (s'(A, B) - s'(D, C)))_k$  d'où, parce que  $a_k \neq 0$ ,  $1 \leq k \leq n$ , on déduit

$$(A, B) R_s (D, C) \Leftrightarrow s(A, B) = s(D, C) \Rightarrow \text{pr}_k (s'(A, B) - s'(D, C)) = 0, \Rightarrow \\ \Rightarrow s'(A, B) = s'(D, C) \Leftrightarrow (A, B) R_{s'} (D, C).$$

Il est naturel donc de noter  $V_n(s)$  par  $V_n$ .

**4. Proposition.** *L'ensemble  $V_n$  peut-être structuré comme espace vectoriel complexe par le transport de la structure de l'espace  $C^n$ .*

En effet, la fonction  $F: V_n \rightarrow C^n$  donnée par  $F(\overrightarrow{AB}) = s(A, B)$  est une bijection car elle est une surjection par définition et  $F$  est une injection car

$$F(\overrightarrow{AB}) = F(\overrightarrow{DC}) \Rightarrow s(A, B) = s(D, C) \Rightarrow (A, B) R_s (D, C) \Rightarrow \overrightarrow{AB} = \overrightarrow{DC}.$$



La bijection  $F$  permet de transporter sur  $V_n$  la structure d'espace vectoriel complexe de  $C^n$  par l'intermédiaire des lois de composition

$$\bar{u} + \bar{v} = F^{-1}(F(\bar{u}) + F(\bar{v})), \quad a\bar{u} = F^{-1}(aF(\bar{u})), \quad a \in C,$$

d'où on déduit

$$(3) \quad F(\bar{u} + \bar{v}) = F(\bar{u}) + F(\bar{v}), \quad F(a\bar{u}) = aF(\bar{u})$$

pour tout triple ordonné  $(a, \bar{u}, \bar{v}) \in C \times V_n^2$ . La vérification des axiomes de la définition de l'espace vectoriel est immédiate et conduit aux conclusions suivantes : le vecteur libre nul  $\bar{0}$  est le vecteur pour qui  $F(\bar{0}) = \theta_{C^n}$  et donc  $\bar{0} = \overrightarrow{MM}$  pour tout point  $M \in \mathcal{A}$  ; le vecteur libre  $-\bar{u}$  l'opposé du vecteur libre  $u = \overrightarrow{AB}$  est le vecteur pour qui  $F(-\bar{u}) = -F(\bar{u})$  c'est-à-dire  $-u = \overrightarrow{BA}$  ;  $F$  est une fonction linéaire en vue des relations (3) et donc un isomorphisme des espaces vectoriels complexes  $V_n$  et  $C^n$ .

Moyennant l'isomorphisme  $F$  on déduit de  $A_1$ ,  $A_5$  et  $A_7$  que :

$$1) \quad \forall (A, B, C) \in \mathcal{A}^3, \quad \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC},$$

$$2) \quad \forall (A, B) \in \mathcal{A}^2, \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA},$$

$$3) \quad \forall \bar{v} \in V_n, \quad \forall A \in \mathcal{A}, \text{ il existe un point unique } B \in \mathcal{A} \text{ tel que } v = \overrightarrow{AB}.$$

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## ASUPRA UNEI DEFINIȚII A SPAȚIILOR GEOMETRICE

(Rezumat)

Se demonstrează caracterul geometric al definiției vectorilor liberi asociați unui spațiu afin, dată în [1], și structurarea ca spațiu vectorial a mulțimii vectorilor liberi asociați unui spațiu afin, prin intermediul unui transport de structură.





# UNE APPLICATION DES SYSTÈMES HOMOGÈNES DE K. YAMAGUTI DANS LA GÉOMÉTRIE DIFFÉRENTIELLE

PAR

ILIE BURDUJAN

1. Dans [3] K. Y a m a g u t i a introduit la notion de système homogène ( $h$ -système) comme étant la structure algébrique induite d'une algèbre de Lie sur une sous-espace vectoriel d'une sous-algèbre de Lie. Le Théorème 2 de [1] donne la construction d'un foncteur covariant de la catégorie  $A$  des algèbres binaires à catégorie  $H.S.$  des  $h$ -systèmes. Cette construction généralise la bienconnue construction fonctorielle de la catégorie  $Ass$  des algèbres associatifs à catégorie  $L$  des algèbres de Lie.

Dans ce qui suite on concrétise cette construction pour une algèbre binaire définie sur le fibré tangent d'une variété différentielle à l'aide d'une et respectivement deux lois de dérivations covariantes.

2. Soit  $M$  une variété différentielle de classe  $C^\infty$  et  $\mathfrak{X}(M)$  le module des champs de vecteurs de classe  $C^\infty$  définis sur  $M$ . Supposons que sur  $\mathfrak{X}(M)$  est définie une loi de dérivation covariante  $\nabla$ . Alors  $\mathfrak{X}(M)$  peut-être organisée comme algèbre binaire sur  $R$  à l'aide de l'opération  $X.Y = \nabla_X Y$ ,  $\forall X, Y \in \mathfrak{X}(M)$ . D'après le Théorème 2 de [1] il est possible d'associer un  $h$ -système à cette algèbre binaire. D'une manière directe on établie les liaisons entre les opérations du  $h$ -système avec les tenseurs de torsion et de courbure de la dérivation  $\nabla$ . On a :

$$X \circ Y = T(X, Y) + [X, Y],$$

$$[X, Y, Z] = R(X, Y)Z - \nabla_{T(X, Y)} Z,$$

$$[X, Y, Z, U] = R(X, Y)(\nabla_Z U) - R(T(X, Y), Z)U - \nabla_{T(X, Y), Z} U - \nabla_Z (R(X, Y)U) - \nabla_{R(X, Y)Z} U + \nabla_{\nabla_{T(X, Y)} Z} U.$$

Alors on a la

**Proposition 1.** *Des relations (III) et (IV) qui sont satisfaites par les opérations du  $h$ -système (voir [3]), associé à la dérivation covariante  $\nabla$  sont équivalents aux relations de Bianki-Padova qui sont satisfaites par les tenseurs de courbure et de torsion associées à  $\nabla$ .*

**Démonstration.** Utilisons les notations

$$\mathfrak{S}f(X, Y, Z) = f(X, Y, Z) + f(Y, Z, X) + f(Z, X, Y),$$

$$\mathfrak{S}g(X, Y, Z, U) = g(X, Y, Z, U) + g(Y, Z, X, U) + g(Z, X, Y, U),$$



$f$  et  $g$  étant des fonctions arbitraires de trois et respectivement quatre variables. Le premier membre de (III) peut-être modifié dans le mode suivant :

$$\begin{aligned} \mathfrak{S}\{(X \circ Y) \circ Z\} + \mathfrak{S}[X, Y, Z] &= \mathfrak{S}T(T(X, Y), Z) + \mathfrak{S}T([X, Y], Z) + \\ &+ \mathfrak{S}[T(X, Y), Z] + \mathfrak{S}[[X, Y], Z] + \mathfrak{S}R(X, Y)Z - \mathfrak{S}\nabla_{T(X, Y)}Z = \\ &= \mathfrak{S}T(T(X, Y), Z) + \mathfrak{S}T(\nabla_Y X, Z) + \mathfrak{S}T(\nabla_X Y, Z) - \mathfrak{S}T(T(X, Y), Z) + \\ &+ \mathfrak{S}[T(X, Y), Z] + \mathfrak{S}R(X, Y)Z - \mathfrak{S}\nabla_Z(T(X, Y)) - \mathfrak{S}[T(X, Y), Z] - \\ &- \mathfrak{S}T(T(X, Y), Z) = \mathfrak{S}R(X, Y)Z - \mathfrak{S}T(T(X, Y), Z) - \mathfrak{S}\nabla_Z(T(X, Y)) + \\ &+ \mathfrak{S}T(\nabla_Z X, Y) + \mathfrak{S}T(X, \nabla_Z Y) = \mathfrak{S}R(X, Y)Z - \mathfrak{S}T(T(X, Y), Z) - \\ &- \mathfrak{S}(\nabla_Z T)(X, Y). \end{aligned}$$

D'ici résulte que (III) est équivalent à la première relation de Bianchi-Padova. Maintenant, on modifie le premier membre de la relation (IV) dans le mode suivant :

$$\begin{aligned} \mathfrak{S}[X \circ Y, Z, U] + \mathfrak{S}[X, Y, Z, U] &= \mathfrak{S}R(T(X, Y), Z)U - \mathfrak{S}\nabla_{T(T(X, Y), Z)}U + \\ &+ \mathfrak{S}R(\nabla_X Y, Z)U - \mathfrak{S}R(\nabla_Y X, Z)U - \mathfrak{S}R(T(X, Y), Z)U - \mathfrak{S}\nabla_{T(\Delta_X Y, Z)}U + \\ &+ \mathfrak{S}\nabla_{T(\Delta_Y X, Z)}U + \mathfrak{S}\nabla_{T(T(X, Y), Z)}U + \mathfrak{S}R(X, Y)(\nabla_Z U) - \mathfrak{S}R(T(X, Y), Z)U - \\ &- \mathfrak{S}\nabla_{R(X, Y)Z}U + \mathfrak{S}\nabla_{T(T(X, Y), Z)}U + \mathfrak{S}\nabla_{\Delta_Z T(X, Y)}U - \mathfrak{S}\nabla_Z(R(X, Y), U) = \\ &= -\mathfrak{S}R(T(X, Y), Z)U - \mathfrak{S}\{\nabla_Z(R(X, Y)U) - R(\nabla_Z X, Y)U - \\ &- R(X, \nabla_Z Y)U - R(X, Y)(\nabla_Z U)\} - \mathfrak{S}\nabla_{R(X, Y)Z}U + \mathfrak{S}\nabla_{T(T(X, Y), Z)}U + \\ &+ \mathfrak{S}\nabla_{\Delta_Z T(X, Y)}U - \mathfrak{S}\nabla_{T(\Delta_Z X, Y)}U + \mathfrak{S}\nabla_{T(X, \Delta_Z Y)}U = -\mathfrak{S}\{R(T(X, Y), Z)U + \\ &+ (\nabla_Z R)(X, Y)U + \nabla_{R(X, Y)Z}U - \nabla_{T(T(X, Y), Z)} + (\Delta_Z T)(X, Y)U\} = \\ &= -\mathfrak{S}\{R(T(X, Y), Z)U + (\nabla_Z R)(X, Y)U\}, \end{aligned}$$

d'où la proposition.

*Remarque* L'assertion de la Proposition 1 peut-être établie aussi pour les variétés munies de quasi-connexions.

3. Soit  $M$  un  $C^\infty$ -variété dont le fibré tangent est muni de deux dérivations covariantes  $\overset{1}{\nabla}$  et  $\overset{2}{\nabla}$ . Le module  $\mathfrak{X}(M)$  peut-être organisé comme algèbre binaire sur  $R$  à l'aide des opérations  $X \circ Y = \overset{1}{\nabla}_X Y$  et  $X \times Y = \overset{2}{\nabla}_X Y$ . Notons par  $\textcircled{1}$ ,  $[\dots]_1$  et respectivement  $\textcircled{2}$ ,  $[\dots]_2$  les opérations des  $h$ -systèmes associées aux dérivations covariantes  $\overset{1}{\nabla}$  et  $\overset{2}{\nabla}$ . Soient  $\overset{1}{R}$ ,  $\overset{2}{R}$ ,  $\overset{1}{T}$  et  $\overset{2}{T}$  les tenseurs de courbure et ces de torsions associés pour les deux dérivations covariantes.  $\mathfrak{X}(M)$  peut être organisé comme algèbre sur  $R$  à l'aide de l'opération  $X \cdot Y = S_X Y$ ,  $S$  étant le tenseur de déformation de la paire de dérivations covariantes. On constate immédiatement :

$$\begin{aligned} X \circ Y &= \overset{2}{T}(X, Y) - \overset{1}{T}(X, Y) = X \textcircled{1} Y - X \textcircled{2} Y, \\ [X, Y, Z] &= [X, Y, Z]_1 + [X, Y, Z]_2 - 2\rho(X, Y)Z + \overset{1}{\nabla}_{T(X, Y)}^2 Z + \overset{2}{\nabla}_{T(X, Y)}^1 Z, \end{aligned}$$



où  $\rho$  est le tenseur de courbure mixte de la paire de dérivations covariantes (voir [2], p. 80).

D'une manière analogue on établit la

**Proposition 2.** *Les relations (III) et (IV) qui sont satisfaites par les opérations du  $h$ -système associé à une paire de dérivations covariantes définies sur le fibré tangent d'une variété différentielle sont équivalentes aux relations de Bianchi-Padova écrites pour la courbure mixte, les courbures et les torsions associées à ces deux dérivations covariantes.*

En utilisant l'induction complète on vérifie la

**Proposition 3.** *Les  $h$ -systèmes associées aux dérivations covariantes  $\nabla^1$  et  $\nabla^2$  sont coïncidents si et seulement si le tenseur de déformation  $S$  de la paire de dérivations  $\nabla^1$  et  $\nabla^2$  satisfait aux conditions*

$$S(X, Y) = S(Y, X),$$

$$S(X, \nabla_Y^1 Z) - S(Y, \nabla_X^1 Z) - \nabla_X^1 S(Y, Z) - \nabla_Y^1 S(X, Z) - S(X, Y \oslash Z) + \\ + S(X, S(Y, Z)) - S(Y, S(X, Z)) = 0,$$

$$[X_1, \dots, X_n, S(Y, Z)]_1 - S(Y, [X_1, \dots, X_n, Z]_1) - S([X_1, \dots, X_n, Y]_1, Z) = 0.$$

*Observation.* La connaissance du  $h$ -système associé à une dérivation covariante est équivalente à la connaissance de cette dérivation si la dérivation est sans torsion.

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#### O APLICAȚIE A SISTEMELOR OMOGENE ÎN SENS K. YAMAGUTI ÎN GEOMETRIA DIFERENȚIALĂ

(Rezumat)

Se concretizează pentru varietăți diferențiabile dotate cu una sau două conexiuni afine, construcția funcțională prezentată în [1]. Se stabilește că axiomele (III) și (IV) satisfăcute de operațiile  $h$ -sistemului unei conexiuni sau unei perechi de conexiuni sînt echivalente cu identitățile lui Bianchi-Padova.

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# QUASI-CONNECTIONS ON MANIFOLDS WITH ALMOST PRODUCT STRUCTURE

BY

ELENA VAMANU

**1. Introduction.** Let  $M$  be a  $C^\infty$ -differentiable,  $n$ -dimensional manifold with an almost product structure defined by tensor field  $F_j^i$  satisfying

$$(1) \quad F_j^i F_k^j = \delta_k^i.$$

If we denote by  $A$  the algebra of real functions on  $M$ , and by  $L$  the module of vector fields on  $M$ , we can give:

**Definition.** A mapping  $D$  that maps each vector field  $X$  of  $L$  onto the linear representation  $D_X$  of  $L$  into itself, is called a *quasi-connection* on  $M$  if it satisfies the following conditions

1.  $D_{fX+gY} = fD_X + gD_Y$ ,  $\forall f, g \in A$ ,  $\forall X, Y \in L$ ;
2.  $D_X fY = fD_X Y + \delta f(X)Y$ ,  $\forall f \in A$ ,  $\forall X, Y \in L$ ;

(this notion is defined in [3], [4]).

We consider the following definitions of the torsion tensor field and the curvature tensor field for a quasi-connection:

$$(2) \quad T(X, Y) = D_X Y - D_Y X - F[FX, FY], \quad \forall X, Y \in L;$$

$$(3) \quad R(X, Y) = D_X D_Y - D_Y D_X - D_{F[FX, FY]}, \quad \forall X, Y \in L,$$

where  $[.]$  denotes the bracket.

Let  $\Gamma_{jk}^i$  be the quasi-connection coefficients, then the local expressions of  $T$  and  $R$  are given respectively by

$$(4) \quad T_{jk}^i = \Gamma_{jk}^i - \Gamma_{kj}^i + F_{h,m}^i F_{[j}^m F_{k]}^h,$$

$$(5) \quad R_{hij}^k = \Gamma_{im}^k \Gamma_{jh}^m - \Gamma_{jm}^k \Gamma_{ih}^m + \Gamma_{jh,m}^k F_i^m - \Gamma_{ih,m}^k F_j^m + \Gamma_{ih}^k F_{m,n}^i F_{[j}^n F_{i]}^m F_{j]}^m$$

(see [3] for details).

The covariant differential of the tensor field  $t_{j_1 j_2 \dots j_p}^{i_1 i_2 \dots i_p}$  with respect to the quasi-connection  $\Gamma_{jk}^i$  is given by

$$(6) \quad D_{kj, j_1 \dots j_q}^{i_1 i_2 \dots i_p} = F_{j, j_1 \dots j_q, m}^{i_1 i_2 \dots i_p} F_k^m + \sum_{\alpha=1}^p \Gamma_{km}^{i\alpha} F_{j, j_1 \dots j_q}^{i_1 i_2 \dots i_{\alpha-1} m i_{\alpha+1} \dots i_p} - \\ - \sum_{\beta=1}^q \Gamma_{kj\beta}^m F_{j_1, j_2 \dots j_{\beta-1} m j_{\beta+1} \dots j_q}^{i_1 i_2 \dots i_p},$$

(see also [3]).

Using (6) we obtain

$$(7) \quad D_k F_j^i = F_{j, m}^i F_k^m + \Gamma_{km}^i F_j^m - \Gamma_{kj}^i F_m^i.$$

When the tensor field  $F$  is covariant constant with respect to the quasi-connection  $\Gamma_{jk}^i$ , ( $D_k F_j^i = 0$ ), we shall call  $\Gamma_{jk}^i$  a  $F$ -quasi-connection.

The problem considered in this paper is that of determining all the  $F$ -quasi-connections of a given  $F$ . Finally we obtain some results on the integrability for an almost product structure in a Riemannian manifold.

**2.  $F$ -quasi-connections.** Let us consider an almost product structure on the differentiable manifold  $\mathcal{M}$  given by  $F$ , and an arbitrary but fixed connection  $\gamma$ .

Let  $\nabla$  denote the covariant differential with respect to  $\gamma$ .

According to M. O b a t a [1], we introduce the following operators :

$$(8) \quad Q_{mh}^{ki} = \frac{1}{2}(\delta_m^k \delta_h^i + F_m^k F_h^i), \quad Q_{mk}^{ki} = \frac{1}{2}(\delta_m^k \delta_h^i - F_m^k F_h^i).$$

These operators act on a general tensor  $T$  according to

$$(9) \quad QT: \quad Q_{mh}^{ki} T \dots^h \dots_k \dots, \quad Q^* T: \quad Q_{mh}^{*ki} T \dots^h \dots_k \dots$$

**Theorem 1.** A necessary and sufficient condition that a quasi-connection  $\Gamma$  in the manifold  $\mathcal{M}$  be a  $F$ -quasi-connection is that  $\Gamma$  can be written in the form

$$(10) \quad \Gamma_{jk}^i = F_j^m (\gamma_{mk}^i + \frac{1}{2} F_h^i \nabla_m F_k^h) + W_{kj}^i,$$

where  $W_{kj}^i$  is an arbitrary tensor meeting condition

$$(11) \quad Q_{mh}^{ki} W_{kj}^h = W_{mj}^i.$$

**Proof.** The covariant differential of the tensor  $F$  with respect to  $\Gamma$  is given by

$$(12) \quad D_k F_j^i = F_{j, m}^i F_k^m + \Gamma_{km}^i F_j^m - \Gamma_{kj}^m F_m^i.$$

The tensor field  $F$  is covariantly constant with respect to  $\Gamma$  if and only if we have

$$(13) \quad D_k F_j^i = 0.$$

If we consider the covariant differential of the tensor field  $F$  with respect to  $\gamma$ , we obtain

$$(14) \quad F_{j, m}^i = \nabla_m F_j^i - \gamma_{mh}^i F_j^h + \gamma_{mj}^h F_h^i.$$

By substituting (14) in (12) we get



$$(15) \quad (\Gamma_{kh}^i - \gamma_{mh}^i F_k^m) F_{kj}^h - (\Gamma_{kj}^h - \gamma_{mj}^h F_k^m) F_h^i = -F_k^m \nabla_m F_j^i.$$

If we denote

$$(16) \quad X_{hk}^i = \Gamma_{kh}^i - \gamma_{mh}^i F_k^m,$$

one can easily prove that  $X$  is a tensor field of type (1, 2). Using (1), (8) and (17) the formula (15) becomes

$$(17) \quad Q_{ph}^{*ji} X_{jk}^h = \frac{1}{2} F_k^m F_j^i \nabla_m F_p^j;$$

(17) is an equation in the unknown tensor  $X$  of type

$$(18) \quad Q^* X = V,$$

with the condition  $QV = 0$ .

The general solution of the equation (18) is given by

$$(19) \quad X = V + W,$$

where  $X$ ,  $V$ ,  $W$  are tensors of the same type, and the arbitrary tensor  $W$  must satisfy the condition (see [1], [5])

$$(20) \quad QW = W.$$

Thus, the general solution of the equation (17) is

$$(21) \quad X_{pk}^i = \frac{1}{2} F_j^i F_k^m \nabla_m F_p^j + W_{pk}^i,$$

with

$$(22) \quad Q_{nm}^{ji} W_{jk}^m = W_{nk}^i.$$

Now, by use of (16) and (21) we obtain

$$(23) \quad \Gamma_{kp}^i = F_k^m (\gamma_{mp}^i + \frac{1}{2} F_j^i \nabla_m F_p^j) + W_{pk}^i.$$

Hence, we have obtained all the  $F$ -quasi-connections.

Let us denote

$$(24) \quad \tilde{\Gamma}_{kp}^i = F_k^m (\gamma_{mp}^i + \frac{1}{2} F_j^i \nabla_m F_p^j).$$

One easily verifies that  $\tilde{\Gamma}_{kp}^i$  defined by (24) is an  $F$ -quasi-connection.

3. Let  $\mathcal{M}$  be a Riemannian manifold,  $\gamma_{jk}^i$  the connection given by Christoffel symbols, and  $F$  the almost product structure defined by

$$(25) \quad F_j^i = \nabla_j v^i,$$

where  $v$  is a vector field on  $\mathcal{M}$  that satisfies certain conditions (see [2] for details).

Let  $T_{jk}^i$  be the torsion tensor of the quasi-connection  $\tilde{\Gamma}_{jk}^i$  and  $N_{jk}^i$  the Nijenhuis tensor of the considered almost product structure, and  $r_{jk}^i$  the curvature tensor of the connection  $\gamma_{jk}^i$ .

It follows by some usual calculations that the Nijenhuis tensor is given by

$$(26) \quad N_{jk}^i = (r_{mkj}^h V^m - 2\overset{\circ}{T}_{jk}^h) \nabla_h V^i.$$

It follows from (26) that  $N_{jk}^i$  vanishes if and only if

$$(27) \quad r_{mkj}^h V^m - 2\overset{\circ}{T}_{jk}^h = 0.$$

Hence, we get the following result:

**Theorem 2.** *The almost product structure defined by the vector field  $v$  is integrable, if and only if the torsion tensor of the quasi-connection  $\overset{\circ}{\nabla}_{jk}^i$  satisfies the condition*

$$(28) \quad \overset{\circ}{T}_{jk}^h = \frac{1}{2} r_{mkj}^h V^m,$$

where  $r_{mkj}^h$  is the curvature tensor of the connection  $\gamma_{jk}^i$ .

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#### CVASI-CONEXIUNI PE VARIETĂȚI CU STRUCTURĂ AFROAPE PRODUS

(Rezumat)

Dată o varietate cu structură aproape produs, în prima parte a articolului se determină toate cvasi-conexiunile în raport cu care tensorul structurii aproape produs este covariant constant. În ultima parte a lucrării se dă o caracterizare pentru integrabilitatea unei anumite structuri aproape produs.



## INVARIANT CURVATURES TO TRANSFORMATIONS OF CONFORMAL METRIC FINSLER CONNECTIONS (II)

BY

I. GHINEA

1. In [1] and [2], by extending a method due to prof. R. Miron [7], we have obtained the general expression of Finsler connections  $(N_j^h, F_{jk}^h, C_{jk}^h)$ , compatible with a conformal metric structure on a differentiable manifold  $M_n$  and also the group of transformations for these connections. Further, in [2], [3], [4] some invariants of that group of transformation have been obtained.

Thus, starting from the  $v$ -tensor of curvature  $S_{jkl}^i$  of the Finsler conformal metric connection  $(N_j^h, F_{jk}^h, C_{jk}^h)$  we get in [4] the conformal  $v$ -tensor of curvature  $W_{jkl}^i$ , given by the formulas [4, (16) and (20)]. We have also established there some properties concerning this invariant.

2. The present paper deals with an other invariant of the group of transformations for conformal metric Finsler connections. We are starting now from the  $h$ -tensor of curvature  $R_{jkl}^i$  and obtain the conformal  $h$ -tensor of curvature of the mentioned Finsler connection.

Let be the formulas [1, (2)] expressing the transformations of conformal metric Finsler connections, when the arbitrary tensors  $A_j^h, \bar{P}_{jk}^h, \bar{Q}_{jk}^h$  are all zero and the connection  $(N_j^h, F_{jk}^h, C_{jk}^h)$  is symmetric :

$$(1) \quad \begin{cases} \bar{N}_j^h = A_j^h, \\ \bar{F}_{jk}^h = F_{jk}^h - \delta_j^h \omega_k - \delta_k^h \omega_j + g_{jk} g^{hm} \omega_m, \\ \bar{C}_{jk}^h = C_{jk}^h - \delta_j^h \lambda_k - \delta_k^h \lambda_j + g_{jk} g^{hm} \lambda_m. \end{cases}$$

We shall consider now the  $h$ -tensor of curvature  $R_{jkl}^h$  of the Finsler conformal metric connection  $(N_j^h, F_{jk}^h, C_{jk}^h)$ , with the components [5, (2)]

$$(2) \quad R_{jkl}^i = \frac{\delta F_{jk}^i}{\delta x^l} - \frac{\delta F_{jl}^i}{\delta x^k} + F_{jk}^h F_{hl}^i - F_{jl}^h F_{hk}^i + C_{jh}^i R_{kl}^h$$

and also the  $h$ -tensor of curvature  $\bar{R}_{jkl}^h$  of the transformed connection  $(\bar{N}_j^h, \bar{F}_{jk}^h, \bar{C}_{jk}^h)$ , with the components

$$(3) \quad \bar{R}_{jkl}^i = \frac{\delta \bar{F}_{jk}^i}{\delta x^l} - \frac{\delta \bar{F}_{jl}^i}{\delta x^k} + \bar{F}_{jk}^h \bar{F}_{hl}^i - \bar{F}_{jl}^h \bar{F}_{hk}^i + \bar{C}_{jh}^i \bar{R}_{kl}^h.$$

If the second relation from (1) is introduced in (3), we get

$$\begin{aligned}
 \bar{R}_{jkl}^i = & \frac{\delta F_{jk}^i}{\delta x^l} - \frac{\delta \omega_k}{\delta x^l} \delta_j^i - \delta_k^i \frac{\delta \omega_j}{\delta x^l} + g_{jk} g^{im} \frac{\delta \omega_m}{\delta x^l} + \omega_m \frac{\delta}{\delta x^l} (g_{jk} g^{im}) - \frac{\delta F_{jl}^i}{\delta x^k} + \\
 (4) \quad & + \delta_j^i \frac{\delta \omega_l}{\delta x^k} + \delta_l^i \frac{\delta \omega_j}{\delta x^k} g_{jl} g^{im} \frac{\delta \omega_m}{\delta x^k} - \omega_m \frac{\delta}{\delta x^k} (g_{jl} g^{im}) + (F_{jk}^h - \delta_j^h \omega_k - \\
 & - \delta_k^h \omega_j + g_{jk} g^{hm} \omega_m) (F_{hl}^i - \delta_h^i \omega_l - \delta_l^i \omega_h + g_{hl} g^{is} \omega_s) - \\
 & - (F_{jl}^h - \delta_j^h \omega_l - \delta_l^h \omega_j + g_{jl} g^{hm} \omega_m) (F_{hk}^i - \delta_h^i \omega_k - \delta_k^i \omega_h + g_{hk} g^{is} \omega_s) + \\
 & + (C_{jh}^i - \delta_j^i \lambda_h - \delta_h^i \lambda_j + g_{jh} g^{im} \lambda_m) R_{kl}^h.
 \end{aligned}$$

But

$$(5) \quad \frac{\delta \omega_k}{\delta x^l} = \omega_{kl} + F_{kl}^m \omega_m,$$

$$(6) \quad \frac{\delta}{\delta x^i} (g_{jk} g^{im}) = F_{jl}^s g_{sk} g^{im} + F_{kl}^s g_{js} g^{im} - F_{sl}^i g_{jk} g^{sm} - F_{sl}^m g_{jk} g^{is}.$$

The equality

$$(7) \quad \bar{R}_{jkl}^i = R_{jkl}^i$$

can be easily verified. Using here the third relation (1) it follows

$$(8) \quad (-\delta_j^i \lambda_h - \delta_h^i \lambda_j + g_{jh} g^{im} \lambda_m) R_{kl}^h = C_{jh}^i R_{kl}^h - C_{jh}^h R_{kl}^i.$$

Let introduce (5) ... (8) into (4); after some calculation we get

$$\begin{aligned}
 \bar{R}_{jkl}^i - C_{jh}^i \bar{R}_{kl}^h = & R_{jkl}^i - C_{jh}^i R_{kl}^h - \delta_j^i (\omega_{kl} + \omega_{lk}) - \delta_k^i \omega_{jl} + \\
 (9) \quad & + \delta_l^i \omega_{jk} + g^{im} (g_{jk} \omega_{ml} - g_{jl} \omega_{mk}) + \delta_l^i \omega_j \omega_k - \delta_l^i g_{jk} g^{hm} \omega_m \omega_h + \\
 & + g_{jk} g^{is} \omega_l \omega_s - \delta_k^i \omega_l \omega_j + \delta_k^i g_{jl} g^{hm} \omega_m \omega_h - g_{jl} g^{is} \omega_k \omega_s.
 \end{aligned}$$

We are now associating to the  $h$ -tensor of curvature  $R_{jkl}^i$ , the tensor  $\mathcal{R}_{jkl}^i$  given by

$$(10) \quad \mathcal{R}_{jkl}^i = R_{jkl}^i - C_{jh}^i R_{kl}^h;$$

this new tensor is antisymmetric in respect to  $k$  and  $l$ .

If we put

$$(11) \quad \omega_{kj} = \omega_{k|j} + \omega_k \omega_j - \frac{1}{2} g_{kj} \omega_m \omega^m = \omega_{k|j} + \omega_k \omega_j - \frac{1}{2} g_{kj} \omega$$

and use (10) and (11), then (9) becomes

$$(12) \quad \bar{\mathcal{R}}_{jkl}^i = \mathcal{R}_{jkl}^i - \delta_j^i (\omega_{kl} - \omega_{lk}) - \delta_k^i \omega_{jl} + \delta_l^i \omega_{jk} + g^{im} (g_{jk} \omega_{ml} - g_{jl} \omega_{mk}).$$

This formula is similar to [4, (8)], obtained there for the  $v$ -tensor of curvature  $S_{jkl}^i$ . Following a same procedure as for  $S_{jkl}^i$ , by successive contractions we shall eliminate  $\omega_{kl}$  and  $\omega_{lk}$  between (12) and its contractions.



We put

$$(13) \quad \begin{aligned} \mathcal{R}_{ikl}^i &= \mathcal{R}_{kl}^i, & \mathcal{R}_{jil}^i &= \mathcal{R}_{jl}^i, & \mathcal{R}_{jki}^i &= \mathcal{R}_{jk}^i, \\ \mathcal{R}_{kl}^i g^{kl} &= \mathcal{R}_i, & \mathcal{R}_{jl}^i g^{jl} &= \mathcal{R}_i, & \mathcal{R}_{jk}^i g^{jk} &= \mathcal{R}_i. \end{aligned}$$

Making in (12)  $i = l$  and summing we obtain

$$(14) \quad \omega_{lk} - \omega_{kl} = \frac{1}{n} (\bar{\mathcal{R}}_{kl} - \mathcal{R}_{kkl}^i).$$

We put again  $i = l$  in (8) and summ, getting

$$(15) \quad \bar{\mathcal{R}}_{jk}^i = \mathcal{R}_{jk}^i - \omega_{kj} + (n-1) \omega_{jk} + g_{jk} \beta,$$

where  $\beta = g^{lm} \omega_{ml}$ .

Now we multiply (15) with  $g^{lk}$  and add. It results  $\beta$  and with this expression of  $\beta$ , (15) leads to

$$(16) \quad (n-1) \omega_{lk} - \omega_{kl} = \left( \bar{\mathcal{R}}_{lk}^i - \frac{1}{2(n-1)} \bar{\mathcal{R}}_i g_{lk} \right) - \left( \mathcal{R}_{lk}^i - \frac{1}{2(n-1)} \mathcal{R}_i g_{lk} \right).$$

The system of two equations (14) and (16), with  $\omega_{lk}$  and  $\omega_{kl}$  as unknowns, gives

$$(17) \quad \begin{cases} \omega_{lk} = \bar{L}_{lk} - L_{lk}, \\ \omega_{kl} = \bar{L}_{lk} - L_{lk} - \frac{1}{n} (\bar{\mathcal{R}}_{kl}^i - \mathcal{R}_{kl}^i), \end{cases}$$

where

$$(18) \quad L_{lk} = \frac{1}{n-2} \left( \mathcal{R}_{lk}^i - \frac{1}{2(n-1)} \bar{\mathcal{R}}_i g_{lk} - \frac{1}{n} \mathcal{R}_{kl}^i \right).$$

Taking into account (17), the relation (12) results in

$$(19) \quad \bar{H}_{jki}^i = H_{jki}^i,$$

where

$$(20) \quad \begin{aligned} H_{jki}^i &= \mathcal{R}_{jki}^i - \frac{1}{n} \delta_j^i \mathcal{R}_{kl}^i + \delta_k^i \left( L_{jl} - \frac{1}{n} \mathcal{R}_{jl}^i \right) - \\ &- \delta_i^i \left( L_{kj} - \frac{1}{n} \mathcal{R}_{jk}^i \right) - g^{lm} \left[ g_{jk} \left( L_{lm} - \frac{1}{n} \mathcal{R}_{ml}^i \right) - g_{jl} \left( L_{km} - \frac{1}{n} \mathcal{R}_{mk}^i \right) \right]. \end{aligned}$$

Thus we obtain

**Theorem 1.** The tensor  $H_{jki}^i$  expressed by (20) is an invariant of the transformation group of the conformal metric Finsler connections (1).

The tensor  $H_{jki}^i$  (20) will be called conformal  $h$ -tensor of curvature.

The invariant  $H_{jkl}^i$  may be written much simpler using the tensors  $\mathcal{R}_{jki}^i$  and  $L_{ik}$ . With this purpose we apply Bianchi's identity [6], [2] for a symmetric connection

$$(21) \quad S_{ijk} \{C_{im}^h R_{jk}^m - R_{ijk}^h\} = 0,$$

where  $S_{ijk} \{\dots\}$  is the symbol of cyclical permutation of  $i, j, k$  and summing. Taking into account (10), relation (21) becomes

$$(22) \quad \mathcal{R}_{jkl}^h + \mathcal{R}_{kij}^h + \mathcal{R}_{lji}^h = 0.$$

We put  $h = j$  in (22) and add :

$$(23) \quad \mathcal{R}_{kl}^j = -(\mathcal{R}_{ki}^j - \mathcal{R}_{ik}^j).$$

From (18) it results that

$$(24) \quad L_{ik} - L_{kl} = \frac{1}{n-2} \left( \mathcal{R}_{ik}^j - \mathcal{R}_{kl}^j - \frac{2}{n-1} \mathcal{R}_{kl}^i \right)$$

and (23), (24) give

$$(25) \quad L_{ik} - \frac{1}{n-1} \mathcal{R}_{kl}^i = L_{kl}.$$

Let introduce the result (25) in the expression (20) for  $H_{jkl}^i$ ; we get the simplified form

$$(26) \quad H_{jkl}^i = \mathcal{R}_{jkl}^i + \delta_j^i (L_{kl} - L_{ik}) + \delta_k^i L_{jl} - \delta_l^i L_{jk} - g^{im} (g_{jk} L_{ml} - g_{jl} L_{mk}).$$

3. Relative to the conformal  $h$ -tensor of curvature  $H_{jkl}^i$  we can prove the following propositions and theorems, similar to those established in [4] regarding the conformal  $v$ -tensor of curvature  $W_{jkl}^i$ .

a) If we take into account that the tensor  $\mathcal{R}_{jkl}^i$  associated to the  $h$ -tensor of curvature  $R_{jkl}^i$  is antisymmetric in the last two indices of covariance  $k$  and  $l$ , we get directly from (26) that

$$(27) \quad H_{jkl}^i = -H_{jlk}^i.$$

It follows thus

**Proposition 1.** *The conformal  $h$ -tensor of curvature  $H_{jkl}^i$  is antisymmetric with respect to the last two indices of covariance  $k$  and  $l$ .*

b) Let consider the contracted tensors of the conformal  $h$ -tensor of curvature  $H_{jkl}^i$ :

$$(28) \quad H_{ikl}^i = H_{kl}, \quad H_{jil}^i = H_{jl}, \quad H_{jki}^i = H_{jk}.$$

Like for the contracted tensors of  $W_{jkl}^i$ , we can prove that

$$(29) \quad H_{kl} = H_{jl} = H_{jk} = 0.$$



We have

**Proposition 2.** *The contracted tensors of the conformal  $h$ -tensor of curvature  $H_{jkl}^i$  are all zero.*

c) Regarding the vanishing of the  $h$ -tensor of curvature  $H_{jkl}^i$  we have the

**Theorem 2.** *The conformal  $h$ -tensor of curvature  $H_{jkl}^i$  is zero, for  $n \geq 4$ , if the following tensor*

$$(30) \quad \mathcal{R}_{jkl}^i = R_{jkl}^i - C_{jh}^i R_{kl}^h$$

is equal to zero.

**Proof.** Suppose  $\mathcal{R}_{jkl}^i = 0$ . It results  $\mathcal{R}_{ikl}^i = \mathcal{R}_{kl} = 0$  and  $\mathcal{R}_{jkl}^i = \mathcal{R}_{jk} = 0$ . With these relations, (18) and (25) give  $\overset{1}{L}_{lk} = L_{kl} = 0$  and (26) results in  $H_{jkl}^i = 0$ .

**Consequences.** The conformal  $h$ -tensor of curvature  $H_{jkl}^i$  is zero in the following cases :

- a)  $F_n$ -spaces having a conformal metric Finsler connection  $(N_j^h, F_{jk}^h, C_{jk}^h)$  with the property that the associated tensor  $\mathcal{R}_{jkl}^i = 0$ .
- b)  $F_n$ -spaces for which  $R_{jkl}^i = 0$  and  $C_{jk}^i = 0$ .
- c)  $F_n$ -spaces with  $R_{jkl}^i = 0$  and  $R_{kl}^i = 0$ .

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## CURBURI INVARIANTE LA TRANSFORMĂRI DE CONEXIUNI FINSLER CONFORM METRICE (II)

(Rezumat)

În prima parte a lucrării [4] s-a determinat  $v$ -tensorul conform de curbura al unei conexiuni Finsler conform metrice. Se stabilește acum existența unui al doilea invariant al grupului de transformări de conexiuni Finsler conform metrice, care se numește  $h$ -tensor conform de curbura; se dau expresiile sale (20) sau (26) și i se studiază o serie de proprietăți.





ON CONFORMAL CURVATURE TENSOR RELATIVE TO THE  
METRIC SEMI-SYMMETRIC CONNECTION

BY

CĂTĂLINA NIȚESCU

Suppose a Riemannian manifold  $V_n$  with metric  $g_{ij}$  admits a metric semi-symmetric connection with connection coefficients  $\Gamma_{jk}^i$  given by

$$\Gamma_{jk}^i = \{j^i_k\} + s_j \delta_k^i - g_{jk} g^{im} s_m,$$

where  $\{j^i_k\}$  are Christoffel symbols of  $V_n$  and  $s_j$  is a covector field.

K. A m u r and S.S. P u j a r prove in [1] that the conformal curvature tensor field is invariant with respect to the transformations

$$(1) \quad \{j^i_k\} \rightarrow \{j^i_k\} + s_j \delta_k^i - g_{jk} g^{im} s_m.$$

In this way one obtains as a corollary the theorem of K. Y a n o [2] which states that a Riemann space  $V_n$  ( $n > 3$ ) can be applied on a space with a metric semi-symmetric connection whose curvature is null if and only if the space  $V_n$  is conformally flat.

In this paper we shall prove that the conformal curvature tensor field is invariant with respect to the transformations

$$(2) \quad \Gamma_{jk}^i \rightarrow \Gamma_{jk}^i + \sigma_j \delta_k^i - g_{jk} g^{im} \sigma_m,$$

where  $\Gamma_{jk}^i$  is a metric semi-symmetric connection, therefore

$$(3) \quad \nabla_k g_{ij} = 0$$

and

$$(4) \quad T_{jk}^i = \frac{1}{n-1} (\delta_j^i T_k - \delta_k^i T_j).$$

It is obvious that the connection

$$(5) \quad \bar{\Gamma}_{jk}^i = \Gamma_{jk}^i + \sigma_j \delta_k^i - g_{jk} g^{im} \sigma_m$$

is also a metric semi-symmetric connection for any covector  $\sigma_j$ .

**Theorem.** *If a differentiable manifold  $M$  admits a metric semi-symmetric connection  $\Gamma_{jk}^i$ , then the conformal curvature tensor field of this connection is equal to the conformal curvature tensor of any other metric semi-symmetric connection (5).*

**Proof 1.** The conformal curvature tensor is invariant [1] with respect to the transformation  $\{j^i_k\} \rightarrow \Gamma^i_{jk}$ , where  $\Gamma^i_{jk}$  is a metric semi-symmetric connection:

$$(6) \quad \Gamma^i_{jk} = \{j^i_k\} + s_j \delta^i_k - g_{jk} g^{im} s_m.$$

Therefore

$$(7) \quad \hat{C}^i_{jkl} = W^i_{jkl},$$

where  $\hat{C}^i_{jkl}$  is the conformal curvature tensor relative to the metric semi-symmetric connection  $\Gamma^i_{jk}$  and  $W^i_{jkl}$  is Weyl's conformal curvature tensor. Substituting (6) into (5) we obtain

$$\bar{\Gamma}^i_{jk} = \{j^i_k\} + p_j \delta^i_k - g_{jk} g^{im} p_m \quad \text{where} \quad p_j = s_j + \sigma_j.$$

We now consider the transformation  $\{j^i_k\} \rightarrow \bar{\Gamma}^i_{jk}$  with respect to which we have

$$(8) \quad \bar{C}^i_{jkl} = W^i_{jkl},$$

where  $\bar{C}^i_{jkl}$  is the conformal curvature tensor relative to the metric semi-symmetric connection  $\bar{\Gamma}^i_{jk}$ . Out of (7) and (8)  $\hat{C}^i_{jkl} = \bar{C}^i_{jkl}$  results and the theorem is proved.

**Proof 2.** The theorem can also be proved directly without using the Proposition 3.1 [1]. Out of

$$\bar{R}^i_{jkl} = \partial_l \bar{\Gamma}^i_{jk} - \partial_k \bar{\Gamma}^i_{jl} + \bar{\Gamma}^s_{jk} \bar{\Gamma}^i_{sl} - \bar{\Gamma}^s_{jl} \bar{\Gamma}^i_{sk}$$

and out of relation (5) we obtain

$$\begin{aligned} \bar{R}^i_{jkl} = & R^i_{jkl} + \delta^i_k \nabla_l \sigma_j - \delta^i_l \nabla_k \sigma_j - g^{im} (g_{jk} \nabla_l \sigma_m - g_{jl} \nabla_k \sigma_m) + \\ & + (\sigma_j \delta^s_k - g_{jk} g^{sp} \sigma_p) (\sigma_s \delta^i_l - g_{sl} g^{iq} \sigma_q) - (\sigma_j \delta^i_l - g_{jl} g^{sp} \sigma_p) (\sigma_s \delta^i_k - \\ & - g_{sk} g^{iq} \sigma_q) + T^i_{kl} \sigma_j - g_{jp} g^{im} \sigma_m T^p_{kl}. \end{aligned}$$

If we note with  $\sigma = g^{qp} \sigma_p \sigma_q$  and if use relation (4) we get

$$\begin{aligned} \bar{R}^i_{jkl} = & R^i_{jkl} + \delta^i_k \left( \nabla_l \sigma_j - \sigma_j \sigma_l + \frac{1}{2} g_{jl} \sigma + \frac{1}{n-1} \sigma_j T_l \right) - \delta^i_l \left( \nabla_k \sigma_j - \right. \\ & - \sigma_j \sigma_k + \frac{1}{2} g_{jk} \sigma + \frac{1}{n-1} \sigma_j T_k \left. \right) - g^{mi} \left\{ g_{jk} \left( \nabla_l \sigma_m - \sigma_m \sigma_l + \frac{1}{2} g_{ml} \sigma + \right. \right. \\ & \left. \left. + \frac{1}{n-1} \sigma_m T_l \right) g_{jl} \left( \nabla_k \sigma_m - \sigma_m \sigma_k + \frac{1}{2} g_{mk} \sigma + \frac{1}{n-1} \sigma_m T_k \right) \right\}. \end{aligned}$$

We now introduce the notation

$$\sigma_{jl} = \nabla_l \sigma_j - \sigma_j \sigma_l + \frac{1}{2} g_{jl} \sigma + \frac{1}{n-1} \sigma_j T_l$$

and the preceding equality becomes



$$(9) \quad \bar{R}_{jkl}^i = R_{jkl}^i + \delta_k^i \sigma_{jl} - \delta_l^i \sigma_{jk} - g^{mt} (g_{jk} \sigma_{mt} - g_{jl} \sigma_{mk}).$$

Contracting  $i$  with  $l$  we get

$$(10) \quad \bar{R}_{jk} = R_{jk} + (2 - n) \sigma_{jk} - g_{jk} \tilde{\sigma}, \quad \text{where} \quad \tilde{\sigma} = g^{mt} \sigma_{mt}$$

and furthermore

$$(11) \quad \bar{R} = R + 2(1 - n) \tilde{\sigma}.$$

Out of (11) results that the  $\tilde{\sigma}$  invariant is

$$(12) \quad \tilde{\sigma} = -\frac{1}{2(n-1)} (\bar{R} - R).$$

Out of (10) for  $n > 2$  we get

$$(13) \quad \sigma_{jk} = -\frac{1}{n-2} \left[ \bar{R}_{jk} - R_{jk} - g_{jk} \frac{\bar{R} - R}{2(n-1)} \right].$$

Substituting (13) in (9) and separating the two categories of terms we get

$$\begin{aligned} \bar{R}_{jkl}^i + \frac{1}{n-2} \left\{ \delta_k^i \bar{R}_{jl} - \delta_l^i \bar{R}_{jk} - g^{mt} (g_{jk} \bar{R}_{ml} - g_{jl} \bar{R}_{mk}) - \frac{\bar{R}}{n-1} (\delta_k^i g_{jl} - \right. \\ \left. - \delta_l^i g_{jk}) \right\} = R_{jkl}^i + \frac{1}{n-2} \{ \delta_k^i R_{jl} - \delta_l^i R_{jk} - g^{mt} (g_{jk} R_{ml} - g_{jl} R_{mk}) - \\ - \frac{R}{n-1} (\delta_k^i g_{jl} - \delta_l^i g_{jk}) \}. \end{aligned}$$

*Remark.* In the second proof we have used the proposition 3.1 [1]. Thus, it will result as a corollary.

**Corollary 1** (K. Amur and S.S. Pujar). *If a Riemannian manifold admits a metric semi-symmetric connection, then the conformal curvature tensor field relative to the metric semi-symmetric connection is equal to the conformal curvature tensor relative to the Riemannian connection.*

*Proof.* The connection  $\{j_k\}$  is a metric semi-symmetric connection with a null  $s_j$  covector.

**Corollary 2.** *If  $\Gamma_{jk}^i$  is a metric semi-symmetric flat connection on a manifold  $\mathcal{M}$  of dimension  $n > 3$  then the tensor  $\check{C}_{jkl}^i$  vanishes.*

**Corollary 3.** *If a manifold  $\mathcal{M}$  of dimension  $n > 3$  admits a metric semi-symmetric connection whose curvature tensor is isotrop then the tensor  $\check{C}_{jkl}^i$  vanishes.*

**Corollary 4.** *If a manifold  $\mathcal{M}$  of dimension  $n > 3$  admits a metric semi-symmetric connection whose Ricci tensor vanishes then the curvature tensor of this connection is equal to the tensor field  $\check{C}_{jkl}^i$ .*

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## ASUPRA TENSORULUI CONFORM DE CURBURĂ RELATIV LA O CONEXIUNE METRICĂ SEMI-SIMETRICĂ

(Rezumat)

Dată fiind o varietate diferențiabilă  $\mathcal{M}$  de dimensiune  $n > 3$  care admite o conexiune metrică semi-simetrică, se demonstrează că tensorul conform de curbura este un invariant al grupului transformărilor de conexiuni metrice semi-simetrice.



APPLICATIONS OF A MANGERON THEOREM FOR TRIANGULAR DOMAINS<sup>1)</sup>

BY

GREGORY M. NIELSON

*Dedicated to Professor D. Mangeron on the 45<sup>th</sup> anniversary of „Mangeron equations“*

1. In the author's paper „A Mangeron Theorem for Triangular Domains“ presented to the SIAM 1976 Fall Meeting [1] the description of a new type of bivariate interpolation method that assumes arbitrary values on the boundary of a triangular domain was exposed. The theory is patterned after analogous results for a rectangular domain which are due to Mangeron [2], and which were already used for different purposes by several scientists (see, for instance, [3]—[9]).

In what follows two examples illustrate two approaches to using the results of our over mentioned research work [1], where all necessary theorems, lemmas, explanations, and introduced notations are to be found.

*Example 1.* Let

$$\begin{aligned} f(x, 0) &= 2 \sin(2\pi x) + x^2 + 1, & f(0, y) &= 4y^2 - 2y + 1, \\ f(x, 1-x) &= \sin^2(2\pi x) - x + 3. \end{aligned}$$

Then (2.10) [1] becomes

$$\begin{aligned} H[f](2x-1) - H[f](x-1) - H[f](x) + H[f](0) &= \\ &= \sin^2(2\pi x) - 2\sin(2\pi x) - 5x(x-1). \end{aligned}$$

From Table 1 we find that

$$H[f](x) = \sin^2(\pi x) + 2\cos^2\left(\frac{\pi x}{2}\right) + 2\sin(\pi x) - \frac{5}{2}(x^2 - 1),$$

which can then be used along with (2.11) [1] to obtain the final interpolant.

It is apparent from the proof of Lemma 2.2 [1] that for the case of polynomials on the boundary, algorithms amenable to computer programs can be developed. These could take the form of either successively com-

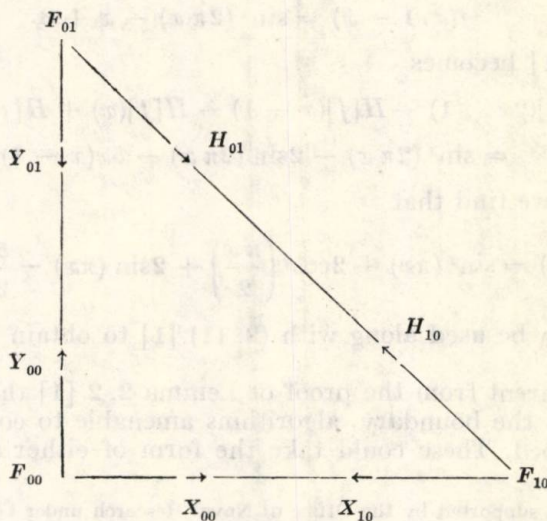
1) Research supported by the Office of Naval Research under Contract NR 044-433.

puting the polynomials  $H[x^n(x-1)]$  in order to complete Table 1 or simply assume a particular basis for  $H[f]$  and determine the unknown coefficients from a system of equations formed from (2.10) [1] by requiring interpolation at the proper number of points.

Table 1

| $f$              | $H(f)$                                                |
|------------------|-------------------------------------------------------|
| $\sin(2\pi x)$   | $-\sin(\pi x)$                                        |
| $\sin^2(2\pi x)$ | $\sin^2(\pi x) + 2\cos^2\left(\frac{\pi x}{2}\right)$ |
| $x(x-1)$         | $\frac{x^2-1}{2}$                                     |
| $x^2(x-1)$       | $\frac{2x+3}{12}(x^2-1)$                              |
| $x^3(x-1)$       | $\frac{6x^2+14x+15}{84}(x^2-1)$                       |
| $x^4(x-1)$       | $\frac{42x^3+135x^2+1127x+180}{1260}(x^2-1)$          |

*Example 2.* In this example, we assume that the function values on the boundary are given as cubic polynomials which are specified by the hermite conditions





$$f(0, 0) = F_{00}, \quad f(1, 0) = F_{10}, \quad f(0, 1) = F_{01}, \quad \partial_x f(0, 0) = X_{00},$$

$$\partial_x f(1, 0) = -X_{10}, \quad \partial_y f(0, 0) = Y_{00}, \quad \partial_y f(0, 1) = -Y_{01};$$

$$(\partial_y - \partial_x) f(1, 0) = H_{10}, \quad (\partial_x - \partial_y) f(0, 1) = H_{01}.$$

In this case, we know that  $H[f]$  of (2.10) will have the form

$$H[f](t) = (t^2 - 1)(\alpha + \beta t)$$

therefore,

$$(1.1) \quad P[f](x, y) = f(x, 0) + f(0, y) - f(0, 0) + [(x - y)^2 - 1](\alpha + \beta(x - y)) - (y^2 - 1)(\alpha - \beta y) - (x^2 - 1)(\alpha + \beta x) - \alpha.$$

As it stands,  $P[f]$  agrees with  $f$  on the two sides  $x = 0$  and  $y = 0$  so all that need be done is to select  $\alpha$  and  $\beta$  so that  $P[f]$  matches the two derivatives along the hypotenuse. This leads to the requirements that

$$2\alpha + 3\beta = \partial_y f(0, 0) - \partial_y f(1, 0),$$

$$2\alpha - 3\beta = \partial_x f(0, 0) - \partial_x f(0, 1).$$

If we solve these equations and make the substitutions into (1.1) along with

$$f(x, 0) = f(0, 0) A_0(x) + \partial_x f(0, 0) B_0(x) + f(1, 0) A_1(x) + \partial_x f(1, 0) B_1(x),$$

$$f(0, y) = f(0, 0) A_0(y) + \partial_y f(0, 0) B_0(y) + f(0, 1) A_1(y) + \partial_y f(0, 1) B_1(y),$$

where  $A_0$ ,  $A_1$ ,  $B_0$  and  $B_1$  are the elementary hermite cubic polynomials; we obtain the nine parameter interpolant

$$\begin{aligned} P[f](x, y) = & F_{00} [A_0(y) - A_1(x)] + F_{10} A_1(x) + F_{01} A_1(y) + \\ & + X_{00} \left[ B_0(x) - \frac{xy}{2} (1 + y - x) \right] - X_{10} \left[ B_1(x) + \frac{xy}{2} (1 - y + x) \right] + \\ & + Y_{00} \left[ B_0(y) - \frac{xy}{2} (1 - y + x) \right] - Y_{01} \left[ B_1(y) + \frac{xy}{2} (1 + y - x) \right] - \\ & - H_{10} \left[ \frac{xy}{2} (y - x - 1) \right] - H_{01} \left[ \frac{xy}{2} (y - x + 1) \right]. \end{aligned}$$

The following Figures 1—6 illustrate the behavior of this interpolant under a variety of different boundary values.

**2. Remark.** One of the intended applications of the author's and other researchers interpolation formulas for triangular domains [10]—[18] is to the very important problem of scattered data interpolation. This is the problem of defining smooth bivariate functions that interpolate to data given on a nonrectangular grid. That is, given  $(x_i, y_i, f_i)$ ,  $i = 1, 2, \dots, n$ ; define  $f$  so that  $f(x_i, y_i) = f_i$ ,  $i = 1, 2, \dots, n^2$ .

<sup>2)</sup> A very excellent survey of this subject has been given by L.L. Schumaker [19].



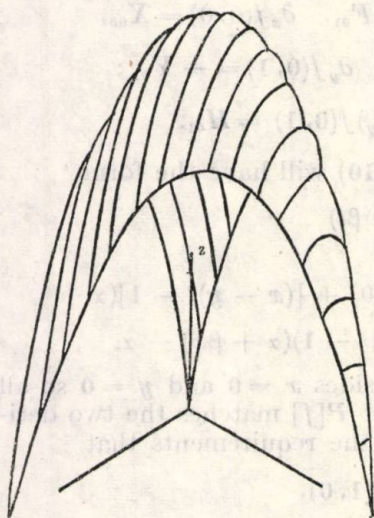


Fig. 1 —  $F_{00} = 0, F_{10} = 0; F_{01} = 0,$   
 $X_{00} = 6, X_{10} = 6, Y_{00} = 6, Y_{01} = 6;$   
 $H_{01} = 6, H_{10} = 6.$

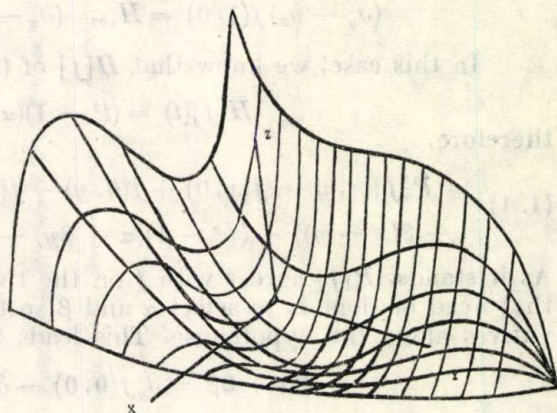


Fig. 2 —  $F_{00} = 1, F_{10} = \frac{1}{2}, F_{01} = 0; X_{00} = -4,$   
 $X_{10} = 4, Y_{00} = -3, Y_{01} = 4; H_{01} = 0, H_{10} = -1$

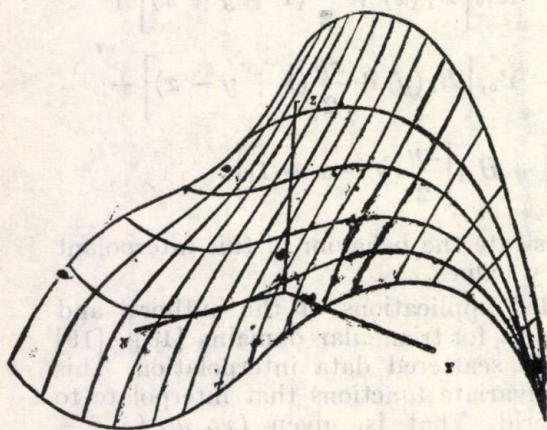


Fig. 3 —  $F_0 = \frac{3}{4}, F_{10} = 0, F_{01} = 0; X_{00} = -2,$   
 $X_{10} = 2, Y_{00} = 1, Y_{01} = 3; H_{01} = 1, H_{10} = -2.$

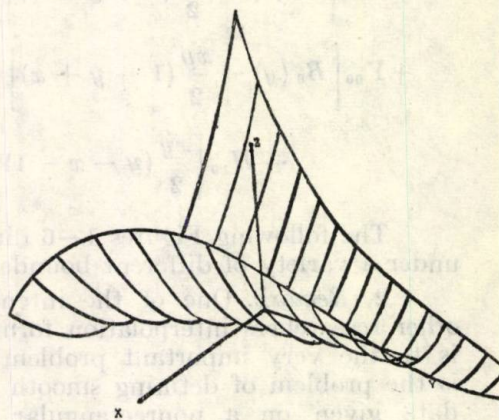


Fig. 4 —  $F_{00} = 1, F_{10} = \frac{1}{2}, F_{01} = 0; X_{00} = -$   
 $X_{10} = -1; Y_{00} = -3, Y_{01} = 1; H_{01} = 0, H_{10} =$



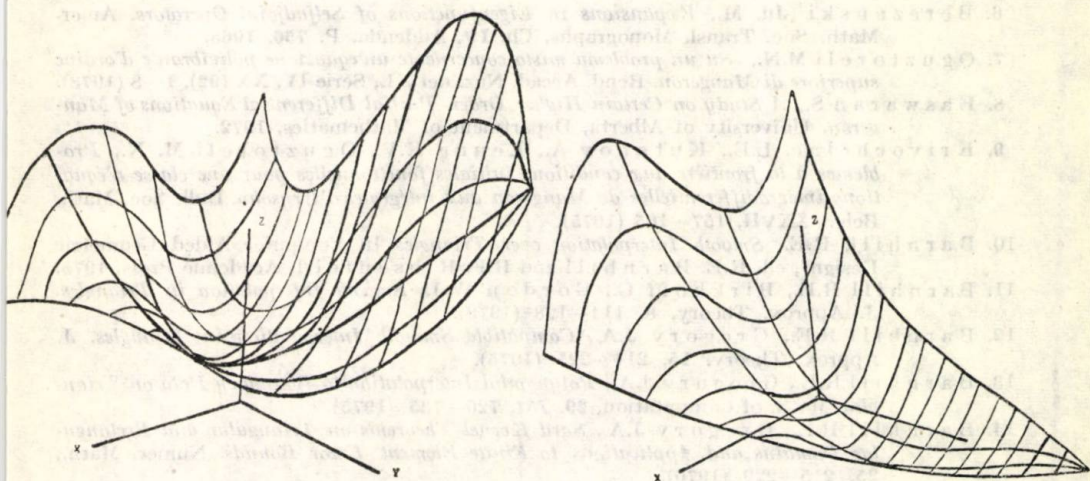


Fig. 5 —  $F_{00}=1, F_{10}=\frac{1}{2}, F_{01}=1; X_{00}=-3, X_{10}=4, Y_{00}=-3, Y_{01}=4, H_{01}=-4, H_{10}=1$ .  
 Fig. 6 —  $F_{00}=1, F_{10}=\frac{1}{2}, F_{01}=0, X_{00}=-3, X_{10}=4, Y_{00}=-3, Y_{01}=1; H_{01}=0, H_{10}=-1$ .

One promising approach is to form a triangularization of the points  $(x_i, y_i)$ ,  $i = 1, 2, \dots, n$  and to piece together functions defined over the resulting triangular regions. In order that this „spline-like“ interpolant have higher order continuity, it is necessary that the basic elements used on each triangle have the property that normal derivatives can be specified on the boundary. Thus it would be very desirable if one could extend the result in [1] for triangular domains so as to include interpolation to normal derivatives. It will be done by using some other Theorems [20]. This would result in interpolation scheme analogous to the Coon's patch for a rectangular domain which has served as the basic element for the blending function methods due to Gordon.

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## APLICAȚII ALE UNEI TEOREME MANGERON LA DOMENII TRIUNGIULARE

(Rezumat)

În [1] s-a expus o metodă de interpolare în două variabile care permite fixarea unor valori arbitrare pe frontiera unui domeniu triunghiular, metodă care folosește rezultate analoge stabilite în [2]. În lucrarea de față se prezintă două exemple care ilustrează rezultatele obținute în [1], unde se găsesc teoremele, lemele, explicațiile și notațiile necesare. Se subliniază apoi posibilitatea de abordare în viitor a problemei de interpolare a datelor împrăștiate arbitrar.



$$(1-\beta) d(x_{p+1}, x_{p+2}) \leq \alpha d(x_p, x_{p+1})$$

et donc

$$d(x_{p+1}, x_{p+2}) \leq \frac{\alpha}{1-\beta} d(x_p, x_{p+1}),$$

où  $\alpha/(1-\beta) < 1$  car  $\alpha + \beta < 1$  de l'hypothèse.

De la même manière, on obtient

$$d(x_{p+2}, x_{p+3}) \leq \frac{\beta}{1-\alpha} d(x_{p+1}, x_{p+2}) \leq \frac{\alpha\beta}{(1-\alpha)(1-\beta)} d(x_p, x_{p+1}).$$

On démontre par induction que

$$d(x_{p+2n}, x_{p+2n+1}) \leq \frac{\alpha^n \beta^n}{(1-\alpha)^n (1-\beta)^n} d(x_p, x_{p+1})$$

et

$$d(x_{p+2n+1}, x_{p+2n+2}) \leq \frac{\alpha^{n+1} \beta^n}{(1-\alpha)^n (1-\beta)^{n+1}} d(x_p, x_{p+1}).$$

En employant l'inégalité du triangle de  $k$  fois, on obtient :

$$\begin{aligned} d(x_{p+n}, x_{p+n+k}) &\leq d(x_{p+n}, x_{p+n+1}) + d(x_{p+n+1}, x_{p+n+2}) + \dots + d(x_{p+n+k-1}, x_{p+n+k}) \leq \\ &\leq \sum_{j=\left[\frac{n}{2}\right]}^{\infty} \frac{\alpha^j \beta^j}{(1-\alpha)^j (1-\beta)^j} d(x_p, x_{p+1}) + \sum_{j=\left[\frac{n}{2}\right]+1}^{\infty} \frac{\alpha^{j+1} \beta^j}{(1-\alpha)^j (1-\beta)^{j+1}} d(x_p, x_{p+1}). \end{aligned}$$

De l'hypothèse pour  $\alpha, \beta$  on a  $\alpha\beta/(1-\alpha)(1-\beta) < 1$  et, par conséquence  $(x_n)_{n \in \mathbb{N}^*}$  est une suite de Cauchy. Soit  $\xi = \lim_{n \rightarrow \infty} x_n$ . Alors

$$\begin{aligned} d(\xi, f(\xi, \xi, \dots, \xi)) &\leq d(\xi, x_{p+2n+1}) + d(x_{p+2n+2}, f(\xi, \xi, \dots, \xi)) \leq \\ &\leq d(\xi, x_{p+2n+2}) + d(g(x_{2n+2}, \dots, x_{p+2n+1}), f(\xi, \xi, \dots, \xi)) = \\ &= d(\xi, x_{p+n+2}) + d(f(\xi, \xi, \dots, \xi), g(x_{2n+2}, x_{p+2n+1})) \leq \\ &\leq d(\xi, x_{p+2n+2}) + \alpha d(\xi, f(\xi, \xi, \dots, \xi)) + \beta d(x_{p+2n+1}, g(x_{2n+2}, \dots, x_{p+2n+1})) \leq \\ &\leq d(\xi, x_{p+2n+2}) + \alpha d(\xi, f(\xi, \xi, \dots, \xi)) + \beta d(x_{p+2n+1}, x_{p+2n+2}). \end{aligned}$$

Donc

$$d(\xi, f(\xi, \xi, \dots, \xi)) \leq \frac{1}{1-\alpha} [d(\xi, x_{p+2n+2}) + \beta d(x_{p+2n+1}, x_{p+2n+2})].$$

Par conséquence, pour  $n \rightarrow \infty$ ,  $d(\xi, f(\xi, \xi, \dots, \xi)) = 0$  où  $\xi = f(\xi, \xi, \dots, \xi)$ . De la même manière

$$d(\xi, g(\xi, \xi, \dots, \xi)) \leq \frac{1}{1-\beta} [d(\xi, x_{p+2n+1}) + \alpha d(x_{p+2n+1}, x_{p+2n+2})]$$



et pour  $n \rightarrow \infty$ ,  $x_{p+2n+1} \rightarrow \xi$  et  $d(x_{p+2n}, x_{p+2n+1}) \rightarrow 0$ , donc  $d(\xi, g(\xi, \xi, \dots, \xi)) = 0$  c'est-à-dire  $\xi$  satisfait l'équation  $\xi = g(\xi, \xi, \dots, \xi)$ .

Pour démontrer l'unicité de l'élément  $\xi$  de  $X$ , on suppose qu'il y a  $\eta$  de  $X$   $\eta \neq \xi$  tel que  $\eta = f(\eta, \eta, \dots, \eta)$  et  $\eta = g(\eta, \eta, \dots, \eta)$ . Parce que  $\xi = f(\xi, \xi, \dots, \xi)$  et  $\xi = g(\xi, \xi, \dots, \xi)$  on obtient

$$d(\xi, \eta) = d(f(\xi, \xi, \dots, \xi), g(\eta, \eta, \dots, \eta)) \leq \alpha d(\xi, f(\xi, \xi, \dots, \xi)) + \beta d(\eta, g(\eta, \eta, \dots, \eta)) = \alpha d(\xi, \xi) + \beta d(\eta, \eta) = 0.$$

Donc  $d(\xi, \eta) = 0$ , c'est-à-dire  $\xi = \eta$ .

**Observation 1.** Si  $f, g$  sont des fonctions d'une seule variable,  $\xi$  est un point fixe commun pour les deux applications  $f, g$ . Ainsi on obtient le Théorème 1. 4, page 97 de [2]. Dans le même cas, pour  $\alpha = \beta < 1/2$ , on obtient le Théorème 1.

## 2. Suites des fonctions définies sur $X^p$ ayant des valeurs en $X$ .

**Théorème 3.** Soient  $(f_n)_{n \in N^*}, (g_n)_{n \in N^*}$  deux suites de fonctions définies sur  $X^p$  ayant des valeurs en  $X$ , où  $(X, d)$  est un espace métrique complet, qui satisfont aux conditions suivantes :

- (i)  $f_n \rightarrow f_0$  uniformément sur  $X^p$  et  $g_n \rightarrow g_0$  uniformément sur  $X^p$  ;
- (ii) il y a les nombres  $\alpha_n, \beta_n \geq 0$ ,  $\alpha_n + \beta_n < 1$ , tels que pour tous  $u_1, u_2, \dots, u_p, v_1, v_2, \dots, v_p$  de  $X$  on a

$$d(f_n(u_1, u_2, \dots, u_p), g_n(v_1, v_2, \dots, v_p)) \leq \alpha_n d(u_p, f_n(u_1, u_2, \dots, u_p)) + \beta_n d(v_p, g_n(v_1, v_2, \dots, v_p))$$

pour  $n = 0, 1, 2, \dots$ ; alors la suite  $(\xi_n)_{n \in N^*}$  de points communs uniques des fonctions  $f_n, g_n$ .

$$(2) \quad \xi_n = f_n(\xi_n, \xi_n, \dots, \xi_n) \quad \text{et} \quad \xi_n = g_n(\xi_n, \xi_n, \dots, \xi_n)$$

converge à  $\xi_0$ , le point commun aux fonctions  $f_0$  et  $g_0$ , où

$$\xi_0 = f_0(\xi_0, \xi_0, \dots, \xi_0) = g_0(\xi_0, \xi_0, \dots, \xi_0).$$

**Démonstration.**  $\xi_n$ , le point qui satisfait aux équations (2), existe et est unique en vertu du Théorème 2. Parce que  $(f_n)_{n \in N^*}$  converge uniformément à  $f_0$  (par l'hypothèse (i)), alors pour tout  $\varepsilon > 0$ , il y a  $n_0(\varepsilon)$  tel que  $n \geq n_0(\varepsilon) \Rightarrow d(f_n(u_1, \dots, u_p), f_0(u_1, \dots, u_p)) < \varepsilon/2$  pour tous  $u_1, u_2, \dots, u_p$  de  $X$ . Mais,

$$\begin{aligned} d(\xi_n, \xi_0) &= d(f_n(\xi_n, \xi_n, \dots, \xi_n), g_0(\xi_0, \xi_0, \dots, \xi_0)) \leq d(f_n(\xi_n, \xi_n, \dots, \xi_n), \\ & f_0(\xi_n, \xi_n, \dots, \xi_n)) + d(f_0(\xi_n, \xi_n, \dots, \xi_n), g_0(\xi_0, \xi_0, \dots, \xi_0)) < \\ & < \frac{\varepsilon}{2} + (d(f_0(\xi_n, \xi_n, \dots, \xi_n), g_0(\xi_0, \xi_0, \dots, \xi_0))). \end{aligned}$$

Cependant

$$\begin{aligned} d(f_0(\xi_n, \xi_n, \dots, \xi_n), g_0(\xi_0, \xi_0, \dots, \xi_0)) &\leq \alpha_0 d(\xi_n, f_0(\xi_n, \xi_n, \dots, \xi_n)) + \\ & + \beta_0 d(\xi_0, g_0(\xi_0, \xi_0, \dots, \xi_0)) = \alpha_0 d(\xi_n, f_0(\xi_n, \xi_n, \dots, \xi_n)), \end{aligned}$$



et, en introduisant cette inégalité dans celle antérieure, on obtient

$$\begin{aligned} d(\xi_n, \xi_0) &< \frac{\varepsilon}{2} + \alpha_0 d(\xi_n, f_0(\xi_n, \xi_n, \dots, \xi_n)) = \\ &= \frac{\varepsilon}{2} + \alpha_0 d(f_n(\xi_n, \xi_n, \dots, \xi_n), f_0(\xi_n, \xi_n, \dots, \xi_n)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Donc  $d(\xi_n, \xi_0) < \varepsilon$  pour  $n \geq n_0(\varepsilon)$ , et l'affirmation du Théorème est démontrée.

**Observation 2.** Dans la démonstration donnée on emploie seulement l'hypothèse que  $f_n \rightarrow f_0$  uniformément et donc la condition  $g_n \rightarrow g_0$  uniformément n'est pas nécessaire. On peut donner une démonstration symétrique en employant la condition que  $g_n \rightarrow g_0$  uniformément sur  $X^p$ .

### 3. Suites récurrentes convergentes.

**Théorème 4.** Soient  $(f_n)_{n \in N^*}$  et  $(g_n)_{n \in N^*}$  deux suites de fonctions définies sur  $X^p$ , ayant des valeurs en  $X$ , où  $(X, d)$  est un espace métrique complet, qui satisfont aux conditions suivantes :

$$\begin{aligned} \text{(i)} \quad & d(f_{n+1}(u_1, u_2, \dots, u_p), f_n(u_1, u_2, \dots, u_p)) \leq a_n, \\ & d(g_{n+1}(u_1, u_2, \dots, u_p), g_n(u_1, u_2, \dots, u_p)) \leq b_n \end{aligned}$$

$$n = 1, 2, \dots, \text{ pour tous } u_1, u_2, \dots, u_p \text{ de } X, \sum_{n=1}^{\infty} a_n < \infty, \sum_{n=1}^{\infty} b_n < \infty;$$

$$\begin{aligned} \text{(ii)} \quad & d(f_n(u_1, u_2, \dots, u_n), g_n(v_1, v_2, \dots, v_p)) \leq \\ & \leq \alpha d(u_n, f_n(u_1, u_2, \dots, u_p)) + \beta d(v_p, g_n(v_1, v_2, \dots, v_p)) \end{aligned}$$

pour tous  $u_1, u_2, \dots, u_p, v_1, v_2, \dots, v_p$  de  $X$  et  $\alpha, \beta > 0, \alpha + \beta < 1$ ;

$$\begin{aligned} \text{(iii)} \quad & x_{p+2n+1} = f_{2n+1}(x_{2n+1}, x_{2n+2}, \dots, x_{p+2n}), \\ & x_{p+2n+2} = g_{2n+2}(x_{2n+2}, x_{2n+3}, \dots, x_{p+2n+1}), \end{aligned}$$

où les éléments  $x_1, x_2, \dots, x_p$  sont arbitraires en  $X$ .

Alors : (i) La suite  $(x_n)_{n \in N^*}$  est convergente en  $X$  à l'élément unique  $\xi_0$  qui satisfait aux équations  $\xi_0 = f_0(\xi_0, \xi_0, \dots, \xi_0)$  et  $\xi_0 = g_0(\xi_0, \xi_0, \dots, \xi_0)$ ;

(ii) les suites  $(f_n)_{n \in N^*}, (g_n)_{n \in N^*}$  convergent uniformément aux fonctions limite  $f_0, g_0$ , qui satisfont à l'inégalité :

$$\begin{aligned} & d(f_0(u_1, u_2, \dots, u_p), g_0(v_1, v_2, \dots, v_p)) \leq \\ & \leq \alpha d(u_p, f_0(u_1, u_2, \dots, u_p)) + \beta d(v_p, g_0(v_1, v_2, \dots, v_p)), \end{aligned}$$

pour tous  $u_1, u_2, \dots, u_p, v_1, v_2, \dots, v_p$  de  $X$ .

La démonstration s'en suit du Théorème précédent et du Théorème 2.

**Observation 3.** Pour montrer que la suite construite dans le Théorème 2 est une suite de Cauchy est suffisant que les fonctions  $f: X^p \rightarrow X$  et  $g: X^p \rightarrow X$ , où  $(X, d)$  est un espace métrique complet, satisfont à l'inégalité moins restrictive

$$d(f(u_1, u_2, \dots, u_p), g(u_2, u_3, \dots, u_{p+1})) \leq \alpha d(u_p, f(u_1, \dots, u_p)) + \beta d(u_2, g(u_2, \dots, u_{p+1}))$$



pour tous  $u_1, u_2, \dots, u_p, u_{p+1}$  de  $X$  et  $\alpha, \beta \in [0, 1)$ ,  $\alpha + \beta < 1$ .

*Observation 4.* La condition du théorème 2, assez restrictive, a été employée seulement pour montrer que  $\xi = \lim_{n \rightarrow \infty} x_n$  est un point fixe pour les fonctions  $f$  et  $g$ . Cette condition a été utilisée aussi pour démontrer l'unicité du point fixe. Si on trouve une condition moins restrictive que celle du théorème 2, on pourrait l'appliquer dans § 2 et § 3.

Reçue le 5 Novembre 1978

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### APLICAȚII ALE UNOR TEOREME DE PUNCTE FIXE COMUNE

(Rezumat)

Plecînd de la teorema lui Kannan — 1968 [1] anume : „dacă  $(X, d)$  este un spațiu metric complet și  $f, g: X \rightarrow X$  două aplicații pentru care există un număr  $\alpha \in (0, 1/2)$  astfel încît are loc (1) pentru orice  $x, y$  din  $X$ , atunci  $f, g$  au un punct fix comun, în paragraful 1 se demonstrează o teoremă de punct fix comun mai generală pentru perechi de funcții definite pe  $X^p$  cu valori în  $X$ , unde  $(X, d)$  este un spațiu metric complet.

În paragraful 2 se studiază punctele fixe comune ale unor șiruri de aplicații definite pe  $X^p$  cu valori în  $X$ , iar în paragraful 3 se obține o teoremă de punct fix pentru șiruri de funcții recurente convergente.

pour tous  $x$  de  $X$ ,  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$ ,  $0 \leq z \leq 1$ .  
 Observation 4. La condition de l'observation 3, assure respectivement à la  
 employée et à son patron pour l'année  $t$  un point fixe pour  
 les fonctions  $F$  et  $G$ . Cette condition a été utilisée pour démontrer  
 l'unicité du point fixe et on trouve une condition moins restrictive que  
 celle de l'observation 3 en posant l'ajout des § 2 et 3.

London, 15 mai 1964  
 C. B. BELLERUP

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## APPENDIX 2. THE CASE OF A SET-VALUED MAPPING

### THEOREM

Let  $X$  be a nonempty compact convex subset of  $\mathbb{R}^n$ . Let  $F: X \rightarrow 2^X$  be a set-valued mapping satisfying the following conditions:  
 (1)  $F(x)$  is nonempty and compact for each  $x \in X$ .  
 (2)  $F(x)$  is convex for each  $x \in X$ .  
 (3)  $F(x)$  is closed for each  $x \in X$ .  
 (4)  $F(x)$  is open for each  $x \in X$ .  
 (5)  $F(x)$  is closed for each  $x \in X$ .  
 (6)  $F(x)$  is open for each  $x \in X$ .  
 (7)  $F(x)$  is closed for each  $x \in X$ .  
 (8)  $F(x)$  is open for each  $x \in X$ .  
 (9)  $F(x)$  is closed for each  $x \in X$ .  
 (10)  $F(x)$  is open for each  $x \in X$ .  
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# NONHOMOGENEOUS EVOLUTION SYSTEMS WITH HOMOGENEOUS INITIAL AND BOUNDARY CONDITIONS

BY  
 N. GRĂDINARU

We find the solution [1] of the system

$$(1) \quad \begin{cases} u_{x_1} - \alpha^2 u_{z_1 z_1} = f_1(x_1, x_2, z_1, z_2), \\ u_{x_2} - \beta^2 u_{z_2 z_2} = f_2(x_1, x_2, z_1, z_2), \end{cases} \quad \begin{cases} 0 < x_1 \leq T, & 0 < z_1 \leq a, \\ 0 < x_2 \leq T, & 0 < z_2 \leq b, \end{cases}$$

in the conditions

$$(2) \quad \begin{cases} u(x_1, x_2, 0, z_2) = 0, \\ u(x_1, x_2, a, z_2) = 0, \\ u(x_1, x_2, z_1, 0) = 0, \\ u(x_1, x_2, z_1, b) = 0, \end{cases} \quad \begin{cases} 0 \leq x_1 \leq T, & 0 \leq x_2 \leq T, \\ 0 \leq z_1 \leq a, & 0 \leq z_2 \leq b; \end{cases}$$

$$(3) \quad u(0, 0, z_1, z_2) = 0, \quad (0 \leq z_1 \leq a, \quad 0 \leq z_2 \leq b),$$

where the partial derivatives  $\frac{\partial f_1}{\partial x_2}, \frac{\partial f_2}{\partial x_1}$  are continuous and satisfy the integrability condition

$$(4) \quad \alpha^2 (f_2)_{z_1 z_1} + f_1 = \beta^2 (f_1)_{z_2 z_2} + f_2.$$

The condition (4) is equivalent to

$$(5) \quad \alpha^2 u_{z_1 z_1 x_1} + (f_1)_{x_1} = \beta^2 u_{z_2 z_2 x_2} + (f_2)_{x_2}.$$

We have:

$$(6) \quad \begin{cases} f_1(x_1, x_2, z_1, z_2) = \sum_{n,m=1}^{\infty} f_{1nm}(x_1, x_2) \sin \frac{n\pi}{a} z_1 \sin \frac{m\pi}{b} z_2, \\ f_2(x_1, x_2, z_1, z_2) = \sum_{n,m=1}^{\infty} f_{2nm}(x_1, x_2) \sin \frac{n\pi}{a} z_1 \sin \frac{m\pi}{b} z_2, \end{cases}$$

where

$$(7) \quad f_{inm}(x_1, x_2) = \frac{4}{ab} \int_0^a \int_0^b f_i(x_1, x_2, s, t) \sin \frac{n\pi}{a} s \sin \frac{m\pi}{b} t \, ds \, dt, \quad (i=1, 2).$$

We seek for a solution of the form

$$(8) \quad u(x_1, x_2, z_1, z_2) = \sum_{n,m=1}^{\infty} u_{nm}(x_1, x_2) \sin \frac{n\pi}{a} z_1 \sin \frac{m\pi}{b} z_2.$$

From (1) (3) and (6)–(8) we obtain the problem

$$(9) \quad \begin{cases} (u_{nm}(x_1, x_2))_{x_1} = -\alpha^2 \left(\frac{n\pi}{a}\right)^2 u_{nm}(x_1, x_2) + f_{1nm}(x_1, x_2), \\ (u_{nm}(x_1, x_2))_{x_1} = -\beta^2 \left(\frac{m\pi}{b}\right)^2 u_{nm}(x_1, x_2) + f_{2nm}(x_1, x_2), \end{cases}$$

$$(10) \quad u_{nm}(0, 0) = 0.$$

Then, we have

$$((u_{nm}(x_1, x_2))_{x_1})_{x_2} = -\beta^2 \left(\frac{m\pi}{b}\right)^2 \left( -\alpha^2 \left(\frac{n\pi}{a}\right)^2 u_{nm}(x_1, x_2) + f_{1nm}(x_1, x_2) \right) + f_{2nm}(x_1, x_2),$$

$$((u_{nm}(x_1, x_2))_{x_1})_{x_1} = -\alpha^2 \left(\frac{n\pi}{a}\right)^2 \left( -\beta^2 \left(\frac{m\pi}{b}\right)^2 u_{nm}(x_1, x_2) + f_{2nm}(x_1, x_2) \right) + f_{1nm}(x_1, x_2)$$

and therefore the integrability for (9) is

$$(11) \quad -\beta^2 \left(\frac{m\pi}{b}\right)^2 f_{1nm} + f_{2nm} = -\alpha^2 \left(\frac{n\pi}{a}\right)^2 f_{2nm} + f_{1nm}.$$

Taking into account (4) and (6) it follows

$$\begin{aligned} & -\alpha^2 \sum_{n,m=1}^{\infty} \left(\frac{n\pi}{a}\right)^2 f_{2nm} \sin \frac{n\pi}{a} z_1 \sin \frac{m\pi}{b} z_2 + \sum_{n,m=1}^{\infty} f_{1nm} \sin \frac{n\pi}{a} z_1 \sin \frac{m\pi}{b} z_2 = \\ & = -\beta^2 \sum_{n,m=1}^{\infty} \left(\frac{m\pi}{b}\right)^2 f_{1nm} \sin \frac{n\pi}{a} z_1 \sin \frac{m\pi}{b} z_2 + \sum_{n,m=1}^{\infty} f_{2nm} \sin \frac{n\pi}{a} z_1 \sin \frac{m\pi}{b} z_2. \end{aligned}$$

Therefore the integrability condition (4) implies (11).

If we put  $u$  instead of  $u_{nm}$  and denote  $k_1 = -\alpha^2 \left(\frac{n\pi}{a}\right)^2$ ,  $k_2 = -\beta^2 \left(\frac{m\pi}{b}\right)^2$  the system (9) becomes

$$(12) \quad \begin{cases} u_{x_1} = k_1 u + f_1, \\ u_{x_2} = k_2 u + f_2, \end{cases} \quad u(0, 0) = 0.$$



From (12) we obtain

$$u(x_1, x_2) = e^{k_1 x_1 + k_2 x_2} \int_0^{x_2} \int_0^{x_1} \left( k_2 f_1(s, t) - \frac{\partial f_1(s, t)}{\partial t} \right) e^{-k_1 s - k_2 t} ds dt + \\ + e^{k_1 x_1 + k_2 x_2} \int_0^{x_2} f_2(x_1, t) e^{-k_1 x_1 - k_2 t} dt + \int_0^{x_1} f_1(s, x_2) e^{k_1(x_1 - s)} ds,$$

or

$$u(x_1, x_2) = \int_0^{x_2} f_2(x_1, t) e^{k_2(x_2 - t)} dt + e^{k_2 x_2} \int_0^{x_1} f_1(s, 0) e^{k_1(x_1 - s)} ds.$$

Indeed,

$$u(x_1, x_2) = e^{k_1 x_1} \left[ C(x_2) + \int_0^{x_1} f_1(s, x_2) e^{-k_1 s} ds \right] = \\ = C(x_2) e^{k_1 x_1} + \int_0^{x_1} f_1(s, x_2) e^{k_1(x_1 - s)} ds,$$

$$\frac{\partial u(x_1, x_2)}{\partial x_2} = C'(x_2) e^{k_1 x_1} + \int_0^{x_1} \frac{\partial f_1(s, x_2)}{\partial x_2} e^{k_1(x_1 - s)} ds = \\ = k_2 C(x_2) e^{k_1 x_1} + k_2 \int_0^{x_1} f_1(s, x_2) e^{k_1(x_1 - s)} ds + f_2(x_1, x_2),$$

and therefore

$$C(x_2) = e^{k_1 x_2} \left\{ C_1 + \int_0^{x_2} \left[ \int_0^{x_1} \left( k_2 f_1(s, t) - \frac{\partial f_1(s, t)}{\partial t} \right) e^{-k_1 s} ds + f_2(x_1, t) e^{-k_1 x_1} \right] e^{-k_2 t} dt \right\}.$$

Then, we have

$$e^{k_1 x_1 + k_2 x_2} \int_0^{x_2} e^{-k_1 s} \left( \int_0^{x_1} \frac{\partial f_1(s, t)}{\partial t} e^{-k_2 t} dt \right) ds = \\ = e^{k_1 x_1 + k_2 x_2} \int_0^{x_1} e^{-k_1 s} \left[ f_1(s, x_2) e^{-k_2 x_2} - f_1(s, 0) + k_2 \int_0^{x_2} f_1(s, t) e^{-k_2 t} dt \right] ds = \\ = \int_0^{x_1} f_1(s, x_2) e^{k_1(x_1 - s)} ds - e^{-k_1 x_1} \int_0^{x_1} f_1(s, 0) e^{k_1(x_1 - s)} ds + k_2 \int_0^{x_1} \int_0^{x_2} f_1(s, t) e^{k_1(x_1 - s) + k_2(x_2 - t)} ds dt.$$

As in [1], it results that the function

$$u(x_1, x_2, z_1, z_2) = \sum_{n,m=1}^{\infty} \left[ \int_0^{x_1} f_{2nm}(x_1, t) e^{-\beta^2 \left(\frac{m\pi}{b}\right)^2 (x_1-t)} dt + \right. \\ \left. + e^{-\beta^2 \left(\frac{m\pi}{b}\right)^2 x_1} \int_0^{z_1} f_{1nm}(s, 0) e^{-\alpha^2 \left(\frac{n\pi}{a}\right)^2 (z_1-s)} ds \right] \sin \frac{n\pi}{a} z_1 \sin \frac{m\pi}{b} z_2$$

is the solution of (1)–(3).

*Remark.* The method can be easily extended for systems of the form

$$\begin{cases} u_{x_1} - au_{z_1 z_1} - bu_{z_2 z_2} = f_1(x_1, x_2, z_1, z_2), \\ u_{x_1} - cu_{z_1 z_1} - du_{z_2 z_2} = f_2(x_1, x_2, z_1, z_2), \end{cases}$$

where  $a, b, c, d$  are positive constants.

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#### SISTEME NEOMOGENE DE EVOLUȚIE CU CONDIȚII OMOGENE LA LIMITĂ ȘI INIȚIALE

(Rezumat)

Se aplică o metodă descrisă în [3] pentru ecuații parabolice uniparametrice și se obține un exemplu concret de semigrup de tipul dat în [1], pentru un sistem bi-parametric de ecuații de evoluție.



# CLASSIFICATION OF INVARIANT SOLUTIONS OF RMHD WITH PLANAR SYMMETRY

BY

GH. PROCOPIUC

**1. Preliminaries.** In a previous paper [1] we saw that the system (S) of equations of radiative magnetohydrodynamics (RMHD) for one-dimensional flow admits an invariance Lie group  $G$ , which gives the possibility to organize the solutions of system (S) as  $H$ -invariant solutions under subgroups of  $G$ .

An important numerical feature of  $H$ -invariant solutions is their rank  $\rho$ , which shows the number of independent variables in the system (S/H) [2].

Among the invariant solutions under different subgroups of a group  $G$ , however, there exists an interdependence. If  $H'$  and  $H''$  are two subgroups of  $G$  of orders  $r'$  and  $r''$ , respectively, and  $H' \subset H''$ , then every  $H''$ -invariant solution of rank  $\rho''$  is also a  $H'$ -invariant solution, but of rank  $\rho'$ . As, generally,  $r' < r''$ , it follows that  $\rho' > \rho''$ . Consequently, the solution considered as  $H'$ -invariant solution is defined by the system (S/H') with a greater number of independent variables than in (S/H''). Both families of  $H'$  and  $H''$ -invariant solutions will coincide if and only if  $r' = r''$ .

Let  $H$  be a subgroup of  $G$  and let  $g$  be a transformation of  $G$ . If the  $H$ -invariant solution  $\Phi$  is subject to the transformation  $g$ , we obtain a new solution  $\Phi_g$  of the system (S). From  $gHg^{-1}\Phi_g = \Phi_g$  it follows that the solution  $\Phi_g$  will be a  $gHg^{-1}$ -invariant solution (of the same rank). The subgroups  $H$  and  $gHg^{-1}$  are said to be *similar*. The invariant solutions of system (S) which do not transform one another by any transformation of  $G$ , are called *essentially distinct solutions*. Therefore, the essentially distinct invariant solutions of a given rank are obtained from subgroups that are not similar.

Consequently, the solution of problem of finding the set of all essentially distinct invariant solutions of a given rank  $\rho$ , consists in enumeration of the set of all subgroups, that are not similar, of given orders of the group  $G$ . For this purpose, L.V. Ovsiannikov in [3] introduced the notion of *optimal system*  $\theta_h$  of subgroups of order  $h$  of the group  $G$ , as being the system of subgroups of  $G$  such that :

- 1°. Any two subgroups in  $\theta_h$  are not similar ;
- 2°. Every subgroup of order  $h$  is similar to the subgroups in  $\theta_h$  ;



3°. The number of arbitrary parameters the subgroups in  $\theta_h$  may depend on, does not decrease by passing to another subgroup similar to it.

For the construction of an optimal system of subgroups of  $G$ , we will use a method given in [4], by means of determining the imprimitivity systems of the group  $G$ . For every manifold belonging to an imprimitivity system, there exists a subgroups  $H$  consisting in transformations which transform any point  $M$  of this manifold into the set of its points. Then, the subgroup  $gHg^{-1}$  will correspond, in this sense, to the manifold belonging to the same imprimitivity system which will contain the point  $M_g$  in which the transformation  $g$  transforms the point  $M$ . In other words, one and the same system of imprimitivity corresponds entirely to the class of subgroups similar to  $H$ . On the other hand, in a simply transitive group, any system of imprimitivity is a complete system of invariants of the reciprocal group, and conversely.

In that follows, we establish the optimal systems of first order subgroups of the groups  $G_8^{11}$  (RMHD 1),  $G_6^{13}$  (RMHD 2), and  $G_6^7$  (RHD 1) [1], for uncoupled equations of radiative magnetohydrodynamics and hydrodynamics of a perfectly transparent fluid.

**2.1. The Optimal System of  $G_8^{11}$  (RMHD 1).** Let  $a^i$  ( $i = \overline{1, 8}$ ) be the coordinates of the parameter group  $G_8$  of the group  $G_8^{11}$  (RMHD 1). Its infinitesimal operators are:

$$(1) \quad \begin{aligned} \chi_1 &= \frac{\partial}{\partial a^1}, \quad \chi_2 = \frac{\partial}{\partial a^2}, \quad \chi_3 = \frac{\partial}{\partial a^3}, \quad \chi_4 = \frac{\partial}{\partial a^4}, \quad \chi_5 = \frac{\partial}{\partial a_5}, \\ \chi_6 &= a^4 \frac{\partial}{\partial a^3} - a^3 \frac{\partial}{\partial a^4} + \frac{\partial}{\partial a^6}, \quad \chi_7 = \frac{\partial}{\partial a^7}, \quad \chi_8 = a^1 \frac{\partial}{\partial a^1} + a^2 \frac{\partial}{\partial a^2} + \frac{\partial}{\partial a^8}. \end{aligned}$$

To find the set of its first order subgroups, we first determine its one-dimensional systems of imprimitivity. Defining the derivatives  $p^k$  ( $k = \overline{1, 7}$ ) by means of relations

$$(2) \quad da^k = p^k da^8,$$

we extend the operators (1) with respect to the variables  $a^k$ :

$$(1') \quad \begin{aligned} \tilde{\chi}_1 &= \chi_1, \quad \tilde{\chi}_2 = \chi_2, \quad \tilde{\chi}_3 = \chi_3, \quad \tilde{\chi}_4 = \chi_4, \quad \tilde{\chi}_5 = \chi_5 \\ \tilde{\chi}_6 &= \chi_6 + p^4 \frac{\partial}{\partial p^3} - p^3 \frac{\partial}{\partial p^4}, \quad \tilde{\chi}_7 = \chi_7, \quad \tilde{\chi}_8 = \chi_8 + p^1 \frac{\partial}{\partial p^1} + p^2 \frac{\partial}{\partial p^2}. \end{aligned}$$

The invariants of the extended group are:

$$(3) \quad \begin{aligned} p^1 e^{-a^8} &= \alpha^1, \quad p^2 e^{-a^8} = \alpha^2, \\ p^3 \cos a^6 - p^4 \sin a^6 &= \alpha^3, \quad p^3 \sin a^6 + p^4 \cos a^6 = \alpha^4, \\ p^5 &= \alpha^5, \quad p^6 = \alpha^6, \quad p^7 = \alpha^7, \end{aligned}$$

where  $\alpha^k$  are constants.



Solving equations (3) with respect to  $p^k$  and substituting in (2), we obtain for the determination of imprimitivity systems

$$(4) \quad \frac{da^1}{\alpha^1 e^{a^1}} = \frac{da^2}{\alpha^2 e^{a^2}} = \frac{da^3}{\alpha^3 \cos a^6 + \alpha^4 \sin a^6} = \frac{da^4}{-\alpha^3 \sin a^6 + \alpha^4 \cos a^6} =$$

$$= \frac{da^5}{\alpha^5} = \frac{da^6}{\alpha^6} = \frac{da^7}{\alpha^7} = \frac{da^8}{\alpha^8},$$

where, for omogeneity, we have introduced the constant  $\alpha^8$ .

Four cases now present themselves, as  $\alpha^8$  and  $\alpha^6$  are, or are not, equal to zero.

a) If  $\alpha^6 \neq 0$  and  $\alpha^8 \neq 0$ , Eqs. (4) give the system of imprimitivity

$$(5) \quad \lambda^1 = \alpha^8 a^1 - \alpha^1 e^{a^1}, \quad \lambda^2 = \alpha^8 a^2 - \alpha^2 e^{a^2},$$

$$\lambda^3 = \alpha^6 a^3 - (\alpha^3 \sin a^6 - \alpha^4 \cos a^6), \quad \lambda^4 = \alpha^6 a^4 - (\alpha^3 \cos a^6 + \alpha^4 \sin a^6),$$

$$\lambda^5 = \alpha^6 a^5 - \alpha^5 a^6, \quad \lambda^6 = \alpha^7 a^6 - \alpha^6 a^7, \quad \lambda^7 = \alpha^8 a^7 - \alpha^7 a^8.$$

The reciprocal group is defined by the operators

$$\bar{\chi}_1 = \alpha^8 \frac{\partial}{\partial \lambda^1}, \quad \bar{\chi}_2 = \alpha^8 \frac{\partial}{\partial \lambda^2}, \quad \bar{\chi}_3 = \alpha^6 \frac{\partial}{\partial \lambda^3}, \quad \bar{\chi}_4 = \alpha^6 \frac{\partial}{\partial \lambda^4}, \quad \bar{\chi}_5 = \alpha^6 \frac{\partial}{\partial \lambda^5},$$

$$\bar{\chi}_6 = \lambda^4 \frac{\partial}{\partial \lambda^3} - \lambda^3 \frac{\partial}{\partial \lambda^4} - \alpha^5 \frac{\partial}{\partial \lambda^5} + \alpha^7 \frac{\partial}{\partial \lambda^6},$$

$$\bar{\chi}_7 = -\alpha^6 \frac{\partial}{\partial \lambda^6} + \alpha^8 \frac{\partial}{\partial \lambda^7}, \quad \bar{\chi}_8 = \lambda^1 \frac{\partial}{\partial \lambda^1} + \lambda^2 \frac{\partial}{\partial \lambda^2} - \alpha^7 \frac{\partial}{\partial \lambda^7}.$$

The condition that  $\lambda^k = \lambda_0^k$  to be a system of invariants for one-parameter subgroups of reciprocal group leads to one of types of similar subgroups

$$(6) \quad \{\alpha^5 \chi_5 + \alpha^6 \chi_6 + \alpha^7 \chi_7 + \alpha^8 \chi_8, \text{ with } \alpha^6 \text{ and } \alpha^8 \neq 0\}.$$

b) If  $\alpha^6 = 0$ ,  $\alpha^8 \neq 0$ , similarly, we obtain

$$(7) \quad \{\alpha^3 \chi_3 + \alpha^5 \chi_5 + \alpha^7 \chi_7 + \chi_8, \alpha^8 \geq 0\}.$$

c) If  $\alpha^6 \neq 0$ ,  $\alpha^8 = 0$ , we have

$$(8) \quad \{\alpha^5 \chi_5 + \chi_6 + \alpha^7 \chi_7\}, \quad \{\cos \varphi \chi_1 + \sin \varphi \chi_2 + \alpha^5 \chi_5 + \chi_6 + \alpha^7 \chi_7, \varphi \in [0, 2\pi]\}.$$

d) If  $\alpha^6 = \alpha^8 = 0$ , but there exists at least one  $\alpha^j \neq 0$  ( $j = 1, 2, 3, 4, 5, 7$ ), we obtain

$$(9) \quad \{\cos \varphi \chi_1 + \sin \varphi \chi_2 + \alpha^3 \chi_3 + \cos \theta \chi_5 + \sin \theta \chi_7, \alpha^3 \geq 0\},$$

$$\{\alpha^3 \chi_3 + \cos \theta \chi_5 + \sin \theta \chi_7, \alpha^3 \geq 0\}, \quad \{\chi_3\},$$

where  $\varphi$  and  $\theta$  are constants,  $\varphi, \theta \in [0, 2\pi]$ .

Therefore, an optimal system of first order subgroups of the group  $G_8$  is determined by the subgroups with the operators (6), (7), (8), (9).

**2.2. The Optimal System of  $G_5^{13}$  (RMHD 2).** The parameter group  $G_5$  of the group  $G_5^{13}$  (RMHD 2) has the infinitesimal operators

$$\begin{aligned} \chi_1 &= \frac{\partial}{\partial a^1}, \quad \chi_2 = \frac{\partial}{\partial a^2}, \\ (10) \quad \chi_3 &= \cos a^5 \frac{\partial}{\partial a^3} + \sin a^5 \frac{\partial}{\partial a^4}, \quad \chi_4 = -\sin a^5 \frac{\partial}{\partial a^3} + \cos a^5 \frac{\partial}{\partial a^4}, \\ \chi_5 &= \frac{\partial}{\partial a^5}. \end{aligned}$$

Defining  $p^k$  ( $k = \overline{1, 4}$ ) by

$$(11) \quad da^k = p^k da^5,$$

the extended operators are

$$(12) \quad \begin{aligned} \tilde{\chi}_1 &= \chi_1, \quad \tilde{\chi}_2 = \chi_2, \quad \tilde{\chi}_5 = \chi_5, \\ \tilde{\chi}_3 &= \chi_3 - \sin a^5 \frac{\partial}{\partial p^3} + \cos a^5 \frac{\partial}{\partial p^4}, \quad \tilde{\chi}_4 = \chi_4 - \cos a^5 \frac{\partial}{\partial p^3} - \sin a^5 \frac{\partial}{\partial p^4}. \end{aligned}$$

Then, Eqs. (11) can be written as

$$\frac{da^1}{\alpha^1} = \frac{da^2}{\alpha^2} = \frac{da^3}{-\alpha^5 a^4 + \alpha^3} = \frac{da^4}{\alpha^5 a^3 + \alpha^4} = \frac{da^5}{\alpha^5},$$

where  $\alpha^i$  ( $i = \overline{1, 5}$ ) are constants.

The optimal system of subgroups is then given by

$$(13) \quad \begin{aligned} &\{\alpha^1 \chi_1 + \alpha^2 \chi_2 + \chi_5\}, \\ &\{\alpha^1 \chi_1 + \alpha^2 \chi_2 + \alpha^3 \chi_3, (\alpha^1)^2 + (\alpha^2)^2 + (\alpha^3)^2 \neq 0, \alpha^3 \geq 0\}. \end{aligned}$$

**2.3. The Optimal System of  $G_6^7$  (RHD 1).** In this case, the infinitesimal operators of parameter group  $G_6$  are

$$(14) \quad \begin{aligned} \chi_1 &= \frac{\partial}{\partial a^1}, \quad \chi_2 = \frac{\partial}{\partial a^2}, \quad \chi_3 = \frac{\partial}{\partial a^3}, \quad \chi_5 = \frac{\partial}{\partial a^5}, \\ \chi_4 &= a^3 \frac{\partial}{\partial a^3} + \frac{\partial}{\partial a^4} + a^5 \frac{\partial}{\partial a^5}, \quad \chi_6 = a^1 \frac{\partial}{\partial a^1} + a^2 \frac{\partial}{\partial a^2} + \frac{\partial}{\partial a^6} \end{aligned}$$

and for the extended group, with  $da^k = p^k da^6$  ( $k = \overline{1, 5}$ ):

$$\begin{aligned} \tilde{\chi}_1 &= \chi_1, \quad \tilde{\chi}_2 = \chi_2, \quad \tilde{\chi}_3 = \chi_3, \quad \tilde{\chi}_5 = \chi_5, \\ \chi_4 &= \chi_4 + p^3 \frac{\partial}{\partial p^3} + p^5 \frac{\partial}{\partial p^5}, \quad \tilde{\chi}_6 = \chi_6 + p^1 \frac{\partial}{\partial p^1} + p^2 \frac{\partial}{\partial p^2}. \end{aligned}$$



The systems of imprimitivity are solutions of

$$\frac{da^1}{\alpha^1 e^{a^6}} = \frac{da^2}{\alpha^2 e^{a^6}} = \frac{da^3}{\alpha^3 e^{a^4}} = \frac{da^4}{\alpha^4} = \frac{da^5}{\alpha^5 e^{a^4}} = \frac{da^6}{\alpha^6}.$$

Any one-parameter subgroup is similar to one of subgroups

$$(15) \quad \begin{aligned} &\{\cos \varphi \chi_1 + \sin \varphi \chi_2\}, \{\cos \varphi \chi_1 + \sin \varphi \chi_2 + \chi_4\}, \\ &\{\cos \theta \chi_3 + \sin \theta \chi_5\}, \{\cos \theta \chi_3 + \sin \theta \chi_5 + \chi_6\}, \\ &\{\cos \psi \chi_4 + \sin \psi \chi_6\}, \{\cos \varphi \chi_1 + \sin \varphi \chi_2 + \cos \theta \chi_3 + \sin \theta \chi_5\}, \end{aligned}$$

where  $\varphi, \theta, \psi$  are arbitrary constants, belonging to  $[0, 2\pi)$ .

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#### CLASIFICAREA SOLUȚIILOR INVARIANTE ALE MHDR CU SIMETRIE PLANĂ

(Rezumat)

Se arată că problema clasificării grupale a soluțiilor invariante ale sistemului (S) de ecuații ale magnetohidrodinamicii radiative (MHDR) revine la problema determinării sistemelor optimale de subgrupuri ale grupului admis de sistemul (S). Se determină apoi sistemele optimale pentru ecuațiile MHDR și HDR unidimensionale.





## ASUPRA PROIECȚIEI ORTOGONALE AXONOMETRICE A UNUI CERC

DE

MARIA BORȘ

1. Scopul acestei Note este de a determina prin metodele geometriei analitice vîrfurile și semiaxe elipsei ( $E$ ) care este proiecția ortogonală axonometrică a unui cerc ( $C$ ) din spațiu.

Procedeu analitic este motivat de considerațiile sintetice expuse în [1] și [2].

2. Considerăm spațiul geometric tridimensional raportat la reperul ortonormat canonic ( $O, \mathbf{i}, \mathbf{j}, \mathbf{k}$ ) față de care se dau: (2.1) planul axonometric ( $P$ ) de ecuație  $x + y + z = a$ ,  $a > 0$ ; (2.2) un plan ( $Q$ ) oarecare care trece printr-un punct fixat  $C(u, v, w)$  și care are ecuația  $p(x - u) + q(y - v) + r(z - w) = 0$ ,  $(p, q, r) \neq (0, 0, 0)$ ; presupunem că planele ( $P$ ) și ( $Q$ ) nu sînt paralele adică există dreapta ( $P \cap Q$ ) de intersecție a acestor plane și pentru care un vector director este  $\mathbf{d} = (q - r)\mathbf{i} + (r - p)\mathbf{j} + (p - q)\mathbf{k}$ ; presupunem că planele ( $P$ ) și ( $Q$ ) nu sînt perpendiculare adică  $p + q + r \neq 0$ ; (2.3) un cerc oarecare ( $C$ ) de centru  $C$  și rază  $R$ , inclus în planul ( $Q$ ) și ale cărui ecuații sînt

$$p(x - u) + q(y - v) + r(z - w) = 0, (x - u)^2 + (y - v)^2 + (z - w)^2 = R^2.$$

3. Proiecția ortogonală axonometrică  $C'$  a punctului  $C$  este intersecția drepte de ecuație vectorială

$$\mathbf{r}(t) = (t + u)\mathbf{i} + (t + v)\mathbf{j} + (t + w)\mathbf{k}, \quad t \in \mathcal{R},$$

cu planul axonometric ( $P$ ); se obține astfel centrul de simetrie  $C'$  al elipsei ( $E$ ) dat de

$$C' \left( \frac{1}{3}(2u - v - w + a), \frac{1}{3}(-u + 2v - w + a), \frac{1}{3}(-u - v + 2w + a) \right).$$

Diametrul ( $A_1 A_2$ ) al cercului ( $C$ ) a cărui proiecție ortogonală axonometrică este diametrul mare ( $A'_1 A'_2$ ) al elipsei ( $E$ ), este inclus în dreapta prin  $C$  și paralelă cu dreapta ( $P \cap Q$ ), care are ecuația vectorială

$$\mathbf{r}(t) = ((q - r)t + u)\mathbf{i} + ((r - p)t + v)\mathbf{j} + ((p - q)t + w)\mathbf{k}, \quad t \in \mathcal{R};$$

intersectînd această dreaptă cu cercul ( $C$ ) obținem punctele diametral opuse

$$A_1(x_1 = u + Rt_0, y_1 = v + Rt_0, z_1 = w + Rt_0),$$

$$A_2(x_2 = u - Rt_0, y_2 = v - Rt_0, z_2 = w - Rt_0),$$

unde

$$t_0 = R[(q - r)^2 + (r - p)^2 + (p - q)^2]^{-\frac{1}{2}}.$$

Atunci vîrfurile  $A'_1$  și  $A'_2$  ale elipsei ( $E$ ) sînt intersecțiunile cu planul axonometric ale normalelor planului axonometric ( $P$ ) care trec prin  $A_1$  și  $A_2$ :

$$A'_1\left(\frac{1}{3}(2x_1 - y_1 - z_1 + a), \frac{1}{3}(-x_1 + 2y_1 - z_1 + a), \frac{1}{3}(-x_1 - y_1 + 2z_1 + a)\right);$$

$$A'_2\left(\frac{1}{3}(2x_2 - y_2 - z_2 + a), \frac{1}{3}(-x_2 + 2y_2 - z_2 + a), \frac{1}{3}(-x_2 - y_2 + 2z_2 + a)\right).$$

Diametrul ( $A_3 A_4$ ) al cercului ( $C$ ) a cărui proiecție ortogonală axonometrică este diametrul mic ( $A'_3 A'_4$ ) al elipsei ( $E$ ), este inclus în dreapta din planul ( $Q$ ) care trece prin punctul  $C$  și este perpendiculară pe dreapta ( $P \cap Q$ ), care are ecuația vectorială

$$\mathbf{r}(t) = ((pq + pr - q^2 - r^2)t + u)\mathbf{i} + ((pq - qr - p^2 + r^2)t + v)\mathbf{j} + ((pr + qr - p^2 - q^2)t + w)\mathbf{k}, \quad t \in \mathcal{R};$$

intersectînd această dreaptă cu cercul ( $C$ ) obținem punctele diametral opuse

$$A_3(x_3 = u + Rt_0, y_3 = v + Rt_0, z_3 = w + Rt_0),$$

$$A_4(x_4 = u - Rt_0, y_4 = v - Rt_0, z_4 = w - Rt_0),$$

unde

$$t_0 = R[(pq + pr - q^2 - r^2)^2 + (pq - qr - p^2 + r^2)^2 + (pr + qr - p^2 - q^2)^2]^{-\frac{1}{2}}.$$

Atunci vîrfurile  $A'_3$  și  $A'_4$  ale elipsei ( $E$ ) sînt intersecțiunile cu planul axonometric ale normalelor planului axonometric ( $P$ ) care trec prin punctele  $A_3$  și  $A_4$ :

$$A'_3\left(\frac{1}{3}(2x_3 - y_3 - z_3 + a), \frac{1}{3}(-x_3 + 2y_3 - z_3 + a), \frac{1}{3}(-x_3 - y_3 + 2z_3 + a)\right),$$

$$A'_4\left(\frac{1}{3}(2x_4 - y_4 - z_4 + a), \frac{1}{3}(-x_4 + 2y_4 - z_4 + a), \frac{1}{3}(-x_4 - y_4 + 2z_4 + a)\right).$$



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## SUR LA PROJECTION ORTHOGONALE AXONOMÉTRIQUE DU CERCLE

(Résumé)

On trouve, par les méthodes de la géométrie analytique, les coordonnées des sommets de l'ellipse qui est la projection orthogonale axonométrique d'un cercle quelconque de l'espace.

RESEARCH REPORTS

1. Report on the results of the research conducted by the author during the period from January 1, 1950, to December 31, 1950. The research was carried out in the laboratory of the author, and the results are presented in the form of a report.

2. A RESEARCH REPORT ON THE RESEARCH CONDUCTED BY THE AUTHOR

On January 1, 1950, the author began his research on the subject of the effect of temperature on the rate of reaction between hydrogen and oxygen. The results of the research are presented in the form of a report.



# PROIECȚIA CENTRALĂ PE $[B_2]$

DE

I. RUSU, MARIA VIAȘU, MARIA BORȘ, CARMEN MARICA  
și CORNELIA EFTIMESCU

**1. Introducere.** Proiecția centrală (conică) în care tabloul este bi-sectorul al doilea  $[B_2]$  este doar secțiunea cu acest plan a conului vizual în vârful căruia  $S(s, s')$  se consideră observatorul. Fiecare rază vizuală este determinată de centrul de proiecție, vârful  $S(s, s')$  și de un punct al spațiului  $A(a, a')$  caracteristic unui obiect dat. Secționarea cu  $[B_2]$  a menționatului con vizual se reduce deci la intersectarea fiecărei raze vizuale cu tabloul indicat în acest caz.

**2. Proiecția centrală — perspectiva — pe  $[B_2]$  a elementelor geometrice și a figurilor geometrice plane.** Fie centrul  $S(s, s')$  și punctul  $A(a, a')$ . Pentru a se găsi proiecția centrală a punctului dat se consideră  $(SA) \cap [B_2] = A_1(a_1, a'_1)$ . Prin urmare  $A_1 \in (SA)$  și tot  $A_1 \in [B_2]$ , ceea ce face ca proiecțiile  $a_1 = a'_1$  să poată fi găsite la intersectarea proiecțiilor razei vizuale. Deci  $(sa) \cap (s'a') = a_1 \equiv a'_1$ .

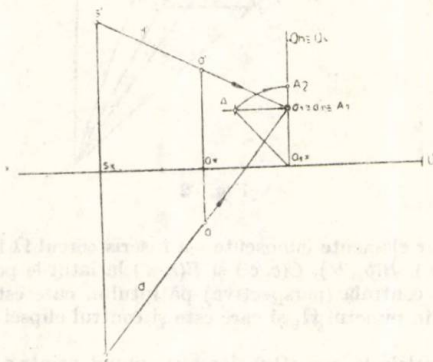


Fig. 1

Se știe că proiecțiile, aici ortogonale, ale obiectelor pe diferite plane nu păstrează, în general, însușirile obiectelor din spațiu privind forma și mărimea, de aceea s-a considerat un plan  $[Q] \perp (Ox)$  ale cărui urme sînt  $(Qh) \equiv (Qv) \perp (Ox)$ .

S-a construit apoi triunghiul de poziție al punctului  $A_1$  din  $[B_2]$  corespunzător diedrului doi în raport cu planul vertical de proiecție și anume cu semiplanul  $[V_s]$ , axa de rabatere fiind axa  $(Ox)$ . Aceasta constituie trecerea la epură în care  $A_1$  ajunge în poziția  $A_2$  (fig. 1).

Din mulțimea figurilor geometrice plane s-a considerat un poligon, pătratul, și o curbă plană, cercul.

a) În fig. 2 se reprezintă pătratul de nivel  $ABCE$  ( $abce, a'b'c'e'$ ) și centrul de proiecție  $S(s, s')$ . Intersectând proiecțiile fiecărei raze vizuale

$$(s, a) \cap (s', a') = A_1(a_1, a'_1); (s, b) \cap (s', b') = B_1(b_1, b'_1);$$

$$(s, c) \cap (s', c') = C_1(c_1, c'_1) \text{ și } (s, e) \cap (s', e') = E_1(e_1, e'_1)$$

se obțin virfurile  $A_1, B_1, C_1, E_1$  ale patrulaterului care este perspectiva pătratului considerat.

b) Se consideră în continuare cercul  $\Omega(\omega, \omega')$  situat într-un plan de front care se proiectează central pe planul  $[B_2]$  cu ajutorul centrului de proiecție  $S(s_1, s'_1)$ .

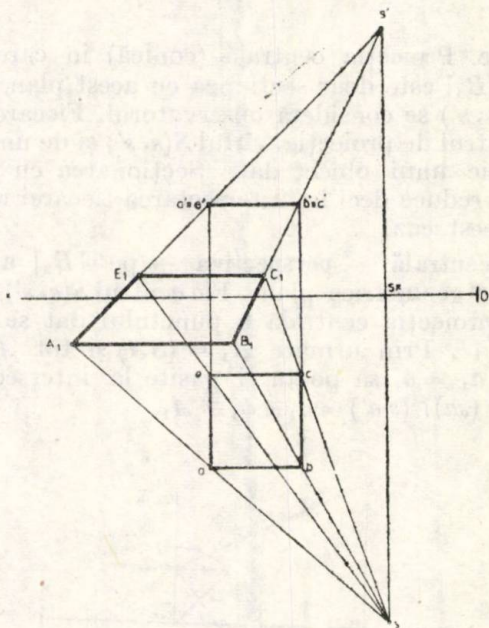


Fig. 2

Pentru utilizarea unor elemente cunoscute s-a înscris cercul  $\Omega$  într-un pătrat, acesta fiind tangent în punctele  $A(a, a')$ ,  $B(b, b')$ ,  $C(c, c')$  și  $E(e, e')$  la laturile poligonului menționat.

Se obține proiecția centrală (perspectiva) pătratului, care este un patrulater, ale cărui diagonale se intersectează în punctul  $\Omega_1$  și care este și centrul elipsei ca proiecție pe  $[B_2]$  a cercului  $\Omega$  considerat.

Ducind prin  $\Omega_1$  paralele la axa  $(Ox)$ , iar apoi unind printr-o dreaptă același centru cu proiecția  $s$  se obțin pe laturile patrulaterului proiecție punctele corespunzătoare  $A_1, B_1, C_1$  și  $E_1$  în care elipsa este tangentă la acesta.

Același rezultat se poate obține, desigur, și intersectând proiecțiile razelor vizuale ale punctelor indicate.

Nefiind suficiente, din punct de vedere grafic, aceste puncte ale cercului, s-au considerat și punctele în care diagonalele pătratului circumscris intersectează cercul  $\Omega$  dat. Astfel punctul  $M(m, m')$  din această categorie are ca proiecție pe  $[B_2]$  punctul  $M_1(m_1, m'_1) = (s, m) \cap (s', m')$ .



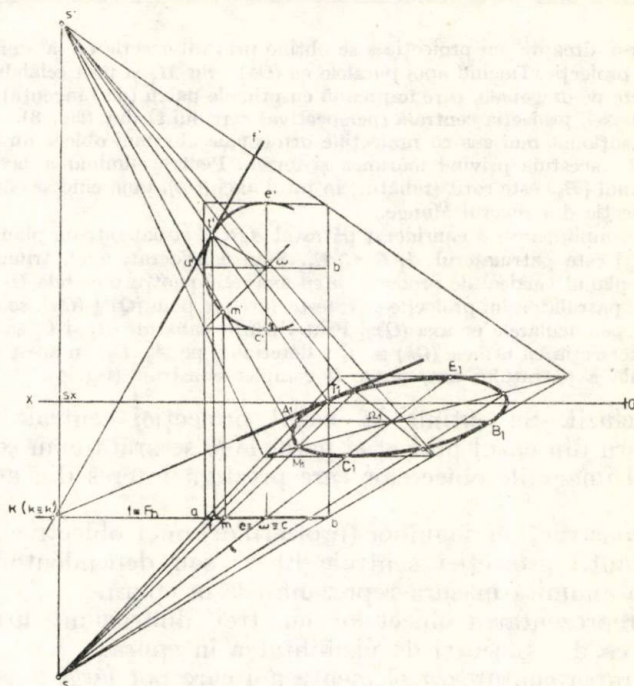


Fig. 3

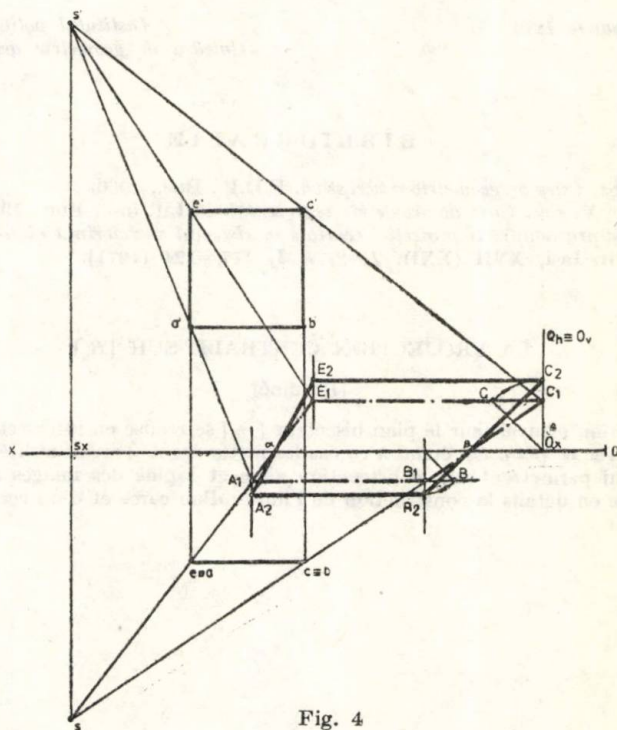


Fig. 4

Unindu-l printr-o dreaptă cu proiecția  $s$  se obține prin intersecție cu a doua diagonală un punct al elipsei proiecție. Ducând apoi paralele cu  $(Ox)$  prin  $M_1$  și prin celălalt punct se găsește încă patru puncte pe diagonale, care împreună cu primele patru (de tangență), permit trasarea cu ușurință a elipsei, proiecția centrală (perspectiva) cercului  $\Omega$  dat (fig. 3).

c) S-a menționat mai sus că proiecțiile ortogonale ale unui obiect nu păstrează, în totalitate, însușirile acestuia privind mărimea și forma. Pentru eliminarea acestui neajuns se consideră că planul  $[B_2]$  este rotit (rabătut) în jurul axei  $(Ox)$ , pînă cînd se confundă cu planul vertical de proiecție din reperul Monge.

Pentru exemplificare s-a considerat pătratul  $ABCE$  situat într-un plan de front a cărui proiecție pe  $[B_2]$  este patrulaterul  $A_1 B_1 C_1 E_1$ . S-au considerat, apoi, triunghiurile de poziție în raport cu planul vertical de proiecție și cu axa  $(Ox)$  pentru punctele  $B_1$  și  $C_1$ . Cunoscînd că fiecare vîrf al patrulaterului proiecție se rotește într-un plan  $[Q] \perp (Ox)$ , se duc și prin punctele  $A_1$  și  $E_1$  perpendicularele pe axa  $(Ox)$ . Prin vîrfurile rabătute  $B_2$  și  $C_2$  se duc paralele la  $(Ox)$ , care la intersecția cu urmele  $(Qh) \equiv (Qv)$  determină pe  $A_2, E_2$ . În acest mod patrulaterul proiecție centrală a pătratului dat, poate fi complet construit (fig. 4).

**3. Concluzii.** Se extind la cazul proiecției centrale pe  $[B_2]$  metodele de lucru din cazul proiecției paralele și se arată cum se pot construi ușor și rapid imaginile obiectelor care prezintă interes din acest punct de vedere.

2. În construcția imaginilor (proiecțiilor) unor obiective se ține seama și de invarianții proiecției centrale libere sau dependente, fapt ce simplifică într-o anumită măsură reprezentările în epură.

3. În reprezentarea obiectelor cu trei dimensiuni urmează să se țină seama, ca de obicei, și de vizibilitatea în epură.

4. Lucrarea conturează elemente noi care pot lărgi domeniul graficii ingineresti.

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#### LA PROJECTION CENTRALE SUR $[B_2]$

(Résumé)

La projection centrale sur le plan bisecteur  $[B_2]$  se réalise en intersectant chaque rayon du cône visuel avec ce plan. On étend à ce cas les méthodes de travail utilisées dans la projection parallèle qui permettent une construction aisée et rapide des images sur ce plan.

On expose en détails la construction de l'image d'un caré et d'un cercle.



# INSTALLATION POUR DÉTERMINER LES PARAMÈTRES MAGNÉTIQUES DE LA COURBE D'HYSTÉRÉSIS EN COURANT ALTERNATIF

PAR

GHIORGHE CĂLUGĂRU, C. COTAE, IRINEL GHEORGHU et EMIL LUCA

**1. Présentation de l'installation.** L'installation qu'on présente représente une installation du type Howling [1] modifiée couplée avec un oscillographe Ferro-Tester (fig. 1).

Les bobines  $B_1$  créent le champ magnétique. Elles sont réalisées sur des carcasses en matière isolateur, leur longueur varie entre 20 et 40 cm. Le diamètre intérieur est de 3—4 cm, mais le diamètre extérieur est déterminé par le nombre des spires bobinées (environ 3 000 spires ayant le diamètre de 1—1,5 mm).

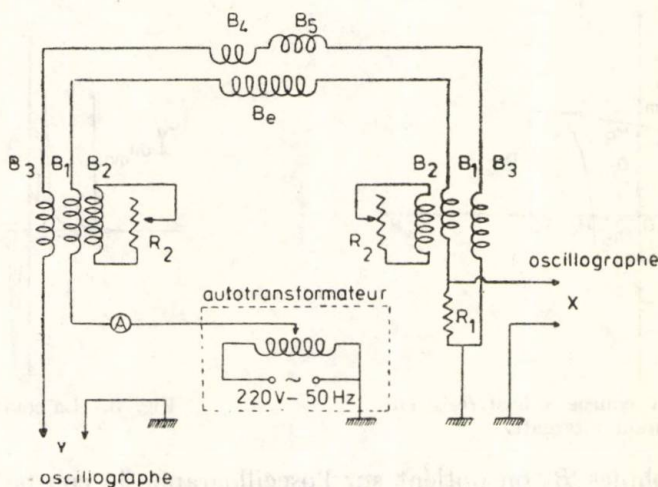


Fig. 1. — Le schéma de l'installation

Les bobines  $B_2$  sont réalisées par l'enroulement d'une couche de fil (Cu) de 1—1,5 mm sur la surface des bobines  $B_1$ . Les résistances  $R_2$  ont des valeurs entre 0—100  $\Omega$ .

La bobine  $B_e$  est une bobine courte (presque 10 cm) avec 200 spires ayant le diamètre de 1—1,5 mm à l'intérieur de laquelle on trouve 2 bobi-

nes  $B_4$  et  $B_5$  liées en opposition (bobinées sur la même carcasse); on a la possibilité de les déplacer par un système mécanique d'une telle manière qu'on puisse faire sortir aussi le système  $B_4-B_5$  en dehors de la bobine  $B_e$ . Les bobines  $B_4-B_5$  contiennent chacune environ 500 spires de fil de 0,1 ou de 0,05 mm.

Les bobines sonde sont réalisées sur des carcasses en matière plastique de la sorte qu'elles entrent dans les bobines  $B_1$ . Leur longueur peut varier entre 5 et 10 cm d'après les nécessités. Elles sont bobinées avec du fil de 0,05 mm ou de 0,1 mm. Le nombre des spires peut varier entre 10 000 et 20 000 spires, d'après la sensibilité de l'installation.

La résistance  $R_1$  dont la valeur la plus grande est d'un ohm doit résister à un courant d'environ 10–15 A.

**2. Réglage de l'installation.** Les bobines  $B_5$  sont liées en opposition. Les tensions récoltées des bobines  $B_3$  sont presque égales et opposées en amplitude et phase et par conséquent elles s'annulent à peu près. La compensation est complètement réalisée par la variation de la résistance  $R_2$  et par le mouvement du système des bobines  $B_4-B_5$  à l'intérieur de la bobine  $B_e$ .

Dans ce cas on obtient sur le Ferro-Tester une ligne horizontale (sans l'épreuve). La facilité avec laquelle les tensions induites dans les bobines peuvent être entièrement annulées fait que l'écrantage ne soit plus nécessaire.

**3. Obtention de la courbe d'hystérésis et de la courbe différentielle.** Par l'introduction dans l'une des bobines  $B_3$  de l'épreuve à étudier sous forme cylindrique ou de bande ayant une longueur beaucoup plus grande que

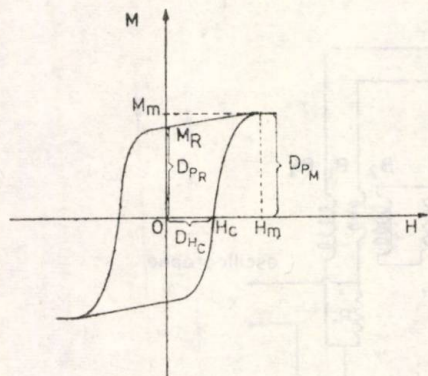


Fig. 2. — La courbe d'hystérésis en courant alternatif.

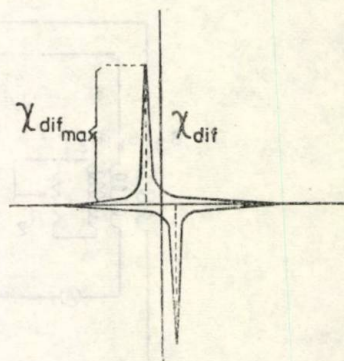


Fig. 3. — La courbe différentielle.

celle des bobines  $B_3$  on obtient sur l'oscillographe la courbe d'hystérésis (fig. 2), quand le commutateur de l'oscillographe est en position d'intégration, ou la courbe différentielle (fig. 3) si le commutateur est en position différentielle (D).

Pour mesurer le champ coercitif et la susceptibilité initiale il est nécessaire de calibrer les bobines du champ  $B_1$  et d'étalonner l'oscillographe sur l'horizontale.



Pour mesurer l'aimantation rémanente  $M_r$  ou de saturation  $M_s$ , il est nécessaire d'étalonner l'oscillographe sur la verticale.

#### 4. Calibrage de l'installation et mesure des caractéristiques magnétiques.

a) *Étallonnage des bobines du champ  $B_1$ .* Avant qu'on fixe les bobines  $B_1$  dans l'installation on introduit un courant continu d'intensité connue  $I_{cn}$  et on mesure à l'aide d'un fluxmètre à bobine, la valeur du champ  $H_{cn}$ . On peut calculer la constante des bobines :

$$(1) \quad C = \frac{H_{cn}}{I_{cn}}.$$

Quand les bobines  $B_1$  sont fixées dans l'installation elles sont alimentées en courant alternatif et le champ maximum  $H_m$  est donné par la relation

$$(2) \quad H_m = CI_m = C \sqrt{2} I_{ef}.$$

Si l'on connaît  $I_{ef}$  mesuré par l'ampermètre ( $A$ ) ainsi que la valeur de  $C$ , on peut calculer la valeur de  $H_m$ .

b) *Étallonnage de l'oscillographe sur l'horizontale et mesure du champ coercitif.* On introduit le courant dans l'installation et on lit la valeur de son intensité sur l'ampermètre ( $A$ ). On obtient une déviation horizontale du spot lumineux  $D_H$ , mesurée à partir du centre de l'écran à droite ou à gauche.

Connaissant, d'après la formule (2) la valeur du champ  $H_m$  correspondant au courant lu sur l'ampermètre ( $A$ ), on peut calculer le champ correspondant à une division de l'écran,  $H_m/D_H$ . Le champ coercitif  $H_c$  de l'épreuve sera donné par la relation :

$$(3) \quad H_c = D_{Hc} \frac{H_m}{D_H},$$

on  $D_{Hc}$  représente le nombre des divisions entre le centre de l'écran ( $O$ ) et le lieu où la courbe coupe l'axe du champ (fig. 2).

c) *Étallonnage de l'oscillographe sur la verticale et mesure de l'aimantation rémanente  $M_r$  et de l'aimantation maxima  $M_m$ .* L'étallonnage peut être fait à l'aide d'une épreuve d'aimantation connue ou à l'aide d'une tension alternative extérieure qui a une valeur connue.

On présente tour à tour les deux méthodes. On introduit une épreuve étalon (on préfère un fil de nickel pur) et on mesure la déviation du spot  $D_{Ni}$  en partant de  $O$  en haut ou en bas. On introduit l'épreuve à mesurer et on mesure le nombre des division  $D_p$  entre lesquelles il y a la relation

$$(4) \quad M_p = M_{Ni} \frac{\gamma_{Ni}^2}{\gamma_p^2} \frac{D_p}{D_{Ni}} = K D_p,$$

ou

$$(5) \quad K = M_{Ni} \frac{\gamma_{Ni}^2}{\gamma_p^2} \frac{1}{D_{Ni}},$$

avec  $r_{Ni}$  — rayon du fil de nickel,  $\gamma_p$  — rayon de l'épreuve cylindrique.



Les longueurs de l'épreuve étalon et de l'épreuve à mesurer doivent être égales et plus grandes que la longueur des bobines sonde  $B_3$ . Utilisant la formule (4) on peut calculer l'aimantation rémanente  $M_R$  et l'aimantation maxima  $M_m$  de l'épreuve à mesurer :

$$(6) \quad M_R = KD_{P_R},$$

où  $D_{P_R}$  représente dans la fig. 2 le nombre des divisions de  $O$  à  $M_R$ , et l'aimantation maxima

$$(7) \quad M_m = KD_{P_m},$$

où  $D_{P_m}$  est présenté dans la fig. 2 et représente le nombre des divisions de  $O$  à  $M_m$ .

On applique potentiométriquement sur la verticale une tension alternative ayant une valeur connue à l'entrée de l'oscillographe Ferro-Tester

$$(8) \quad u_c = U_{cm} \sin \omega t.$$

On obtient sur l'oscillographe, sur l'intégralle une déviation  $l_c$  du spot lumineux, mesurée à partir du centre de l'écran en haut ou en bas. En intégrant la relation (8) on obtient un flux

$$(9) \quad \Phi_c = \int u_c dt = \int U_{cm} \sin \omega t dt = -\frac{U_{cm}}{\omega} \cos \omega t,$$

dont la valeur maxima

$$(10) \quad \Phi_{cm} = \frac{U_{cm}}{\omega}$$

est proportionnelle à la déviation  $l_c$ .

Quand on introduit dans la bobine sonde l'épreuve à mesurer il y a une tension qui apparaît et qui par intégration donne naissance à un flux ayant la valeur maxima  $\Phi_{xm}$  qui produit sur l'oscillographe une déviation  $l_x$ .

Entre ces flux et déviations il y a la relation

$$(11) \quad \Phi_{xm} = \Phi_{cm} \frac{l_x}{l_c} = \frac{U_{cm}}{\omega l_c} l_x.$$

On calcule ensuite la valeur du flux  $\Phi_{xm}$ , ou en général  $\Phi_x$ , qui apparaît ou cas où l'on introduit dans la bobine l'épreuve d'aimantation maxima  $M_m$ , ou en général  $M$ .

Si  $N$  représente le nombre des spires sur chaque bobine  $B_3$ , si  $s$  représente la section de la bobine  $B_3$  et  $s_1$  la section de l'épreuve, la présence de l'épreuve dans l'une des bobines produit une variation de flux d'un bout à l'autre du circuit de bobine  $B_3$ , variation donnée par la relation

$$(12) \quad \begin{aligned} \Phi_x &= \Phi_2 - \Phi_1 = N\mu_0 H(s - s_1) + \mu NHs_1 - \mu_0 NHs = \\ &= Ns_1(\mu - \mu_0)H. \end{aligned}$$



En utilisant la relation

$$(13) \quad B = \mu H = \mu_0 (H + M)$$

on obtient pour  $\Phi_x$  l'expression .

$$(14) \quad \Phi_x = NS_1 \mu_0 M$$

et pour  $\Phi_{xm}$  la relation

$$(15) \quad \Phi_{xm} = NS_1 \mu_0 M_m.$$

Si l'on remplace l'expression (15) dans (11) on obtient

$$(16) \quad M_m = \frac{U_{cm}}{\omega l_c NS_1 \mu_0} l_x,$$

qui peut se présenter aussi sous la forme

$$(17) \quad M_m = \frac{U_{cm}}{\omega l_c N \mu_0 s_2} l_x = K_1 \frac{l_x}{s_1}.$$

Connaissant la valeur de la constante  $K_1$  on peut calculer à l'aide de la relation (17) l'aimantation maxima  $M_m$  ou l'aimantation rémanente  $M_R$  si l'on mesure les déviations  $l_x$  correspondantes et la section de l'épreuve.

d) *Mesurage de la susceptibilité initiale  $\chi_i$ .* Par définition la susceptibilité initiale est

$$(18) \quad \chi_i = \lim_{\substack{\Delta H \rightarrow 0 \\ H \rightarrow 0}} \frac{\Delta M}{\Delta H},$$

done, pour là mesurer il faut calculer le rapport  $\Delta M/\Delta H$  dans le domaine des champs magnétiques faibles. On procède de la manière suivante. On réduit le champ magnétique  $H$  de l'autotransformateur jusqu'à ce que la

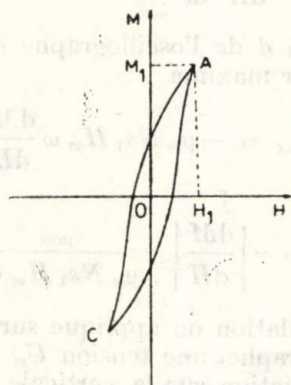


Fig. 4. — Le cycle mineur.

courbe d'hystéresis passe dans un cycle mineur qui peut avoir aussi la forme d'une ligne droite réunissant les points A et C. On détermine  $M_1$  d'après une relation de type (7) ainsi que

$$(19) \quad M_1 = K D_{M_1},$$

où  $D_{M_1}$  représente le nombre des divisions de  $O$  à  $M_1$  (fig. 4). On détermine  $M_1$  en utilisant la relation (2) où  $I_{ef}$  représente le courant sur l'ampermètre (A) quand sur l'oscillographe apparaît la fig. 4.

On détermine la susceptibilité initiale  $\chi_i$  par la relation

$$(20) \quad \chi_i = \frac{M_1}{H_1}.$$

e) *Mesure de la susceptibilité totale  $\chi_T$  à une valeur déterminée du champ  $H$ .*

On procède de la même manière que pour mesurer la susceptibilité initiale avec cette différence qu'on mesure l'aimantation  $M$  pour un champ  $H$  quelconque entre 0 et  $H_m$ :

$$(21) \quad \chi_T = \frac{M}{H}.$$

On obtient la valeur maxima de la susceptibilité en traçant la courbe  $\chi_T = f(H)$  et en prenant la valeur maxima de la courbe.

f) *Mesure de la susceptibilité différentielle  $\chi_{dif}$ .* Par définition

$$(22) \quad \chi_{dif} = \lim_{\Delta H \rightarrow 0} \frac{\Delta M}{\Delta H}.$$

Pour mesurer la susceptibilité initiale il faut tout d'abord calculer la tension totale induite par l'introduction de l'éprouve dans l'une des bobines  $B_1$  où il y a une variation de flux donnée par la formule (14) de la manière que

$$(23) \quad e = - \frac{d\Phi_x}{dt} = - \frac{d\Phi_x}{dH} \frac{dH}{dt} = - N s_1 \mu_0 \frac{dM}{dH} H_m \omega \cos \omega t,$$

qui produit une déviation  $d$  de l'oscillographe sur la position dérivée  $D$ . Cette déviation a la valeur maxima

$$(24) \quad e_{\max} = - \mu_0 N s_1 H_m \omega \frac{dM}{dH},$$

d'où l'on obtient

$$(25) \quad \chi_{dif} = \left| \frac{dM}{dH} \right| = \frac{e_{\max}}{\mu_0 N s_1 H_m \omega}.$$

Pour calibrer l'installation on applique sur les plaques verticales sur la position  $D$  de l'oscillographe, une tension  $U_{ef}$  ayant une valeur maxima  $U_m$  qui produira une déviation sur la verticale  $d_c$ , mesurée à partir de  $O$  en haut ou en bas:

$$(26) \quad d_c = P U_m = \sqrt{2} U_{ef} P,$$

où  $P$  représente la sensibilité de l'oscillographe.

En outre, la valeur  $e_{\max}$  donnée par la relation (24) produit une déviation  $d_x$  inconnue



$$(27) \quad e_{\max} = \frac{1}{P} d_x$$

et

$$(28) \quad \chi_{\text{dif}} = \left| \frac{dM}{dH} \right| = \frac{d_x}{P \eta_0 N s_1 H_m \omega}.$$

En utilisant la relation (26) on obtient

$$(29) \quad \chi_{\text{dif}} = A \frac{d_x}{H_m s_1},$$

avec

$$(30) \quad A = \frac{\sqrt{2} U_{\text{ef}}}{\omega_0 d_c N \omega}.$$

On peut calculer  $\chi_{\text{dif}}$  à l'aide de la relation (29) si l'on connaît la déviation  $d_x$ , le champ maximum et la section de l'épreuve.

**5. Conclusions.** 1. L'étude présente une installation de type Howling modifiée, à l'aide de laquelle on peut tracer le cycle d'hystérésis en régime dynamique pour des champs intenses (jusqu'à  $12 \cdot 10^4$  A/m).

2. On y a montré également la manière dont on calibre l'installation couplée à un Ferro-Tester pour mesurer l'aimantation maxima, l'aimantation rémanente, le champ coercitif, la susceptibilité initiale, totale et différentielle.

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Institut Polytechnique, Jassy,  
et  
Centre de Physique Technique, Jassy

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#### INSTALAȚIE PENTRU DETERMINAREA PARAMETRILOR MAGNETICI DIN CURBA DE HISTEREZIS ÎN CURENT ALTERNATIV

(Rezumat)

Se descrie o instalație utilizată pentru trasarea ciclului de histerezis în regim dinamic la cimpuri magnetice intense ( $12 \cdot 10^4$  A/m) și se prezintă modul de calibrare a instalației pentru măsurarea caracteristicilor magnetice ce decurg din curba de histerezis.

(27)

$$I = \frac{W}{V}$$

(28)

$$I = \frac{W}{V} \left( \frac{1}{1 + \frac{W}{V}} \right)$$

(29)

The relation (29) can be written

(30)

$$I = \frac{W}{V} \left( \frac{1}{1 + \frac{W}{V}} \right)$$

(31)

$$I = \frac{W}{V} \left( \frac{1}{1 + \frac{W}{V}} \right)$$

(32)

$$I = \frac{W}{V} \left( \frac{1}{1 + \frac{W}{V}} \right)$$

On peut calculer  $I$  à l'aide de la relation (29) si l'on connaît la dérivée de la charge maximum et la section de l'épave.

6. Conclusion. 1. L'étude présentée une installation de type Howling modifiée à l'aide de laquelle on peut tracer le cycle d'oscillation en régime dynamique pour des charges intenses (jusqu'à  $12 \cdot 10^4$  A/m).

2. On y a montré également la manière dont on réalise l'installation comprise à un Electro-Tester pour mesurer l'aimantation maximale l'aimantation rémanente, le champ coercitif, la susceptibilité initiale et différentielle.

Centre de Recherches Technologiques, Jassy  
Institut Polytechnique, Jassy

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(Received)

On a étudié l'installation, décrite dans la note précédente, afin de réaliser des mesures de l'aimantation maximale, de l'aimantation rémanente, du champ coercitif et de la susceptibilité initiale et différentielle pour des charges intenses (jusqu'à  $12 \cdot 10^4$  A/m) et de montrer la manière dont on réalise l'installation comprise à un Electro-Tester pour mesurer l'aimantation maximale, l'aimantation rémanente, le champ coercitif, la susceptibilité initiale et différentielle.



INITIAL PERMEABILITY AND CURIE TEMPERATURE IN THE COMPOUNDS OF THE  $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$  SYSTEM

BY

ANASTASIA VASILIU

The introduction of calcium in some spinel ferrites, in stoichiometry conditions or above such conditions, has permitted to obtain some mixt calcium and ferritic systems, with magnetic properties different from those of basic spinel ferrites. Thus, Gorter [1] obtaining the  $\text{Ca}_x\text{Zn}_{1-x}\text{Fe}_2\text{O}_4$  system, where  $x$  has values  $0.2 < x < 0.5$ , notices that in the case of fast cooling, ferrite sinterized at  $1300^\circ\text{C}$  is spinel, monophasic up to a value  $x \approx 0.4$ , while in the case of slow cooling, the formed ferritic system is monophasic only up to values  $x \approx 0.2$ . For samples prepared beyond these values, the obtained ferrite system is pluriphase.

Gorter notices that the highest magnetization at saturation ( $5.3 \mu_B$ ) and the highest Curie temperature ( $T_c = 300^\circ\text{C}$ ) correspond to a system  $\text{Ca}_{0.35}\text{Zn}_{0.65}\text{Fe}_2\text{O}_4$ . He tries to account for it by considering that  $\text{Ca}^{2+}$  ions hold only octahedric places (B). Snek [2] tried to interpret this high value of magnetization at saturation of the system  $\text{Ca}_{0.35}\text{Zn}_{0.65}\text{Fe}_2\text{O}_4$ , which is higher than that of the system  $\text{Ni}_{0.35}\text{Zn}_{0.65}\text{Fe}_2\text{O}_4$  ( $4.8 \mu_B$ ) also containing nickel ions carriers of a magnetic moment. To this end, he yields a nickel ferrite containing also cadmium, viz. the  $\text{Cd}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$  system, where the cadmium ion with big ionic ray ( $1.02 \text{ \AA}$ ), as well as the calcium ion ( $1.06 \text{ \AA}$ ) substitutes ions with smaller ion ray — Ni ( $0.78 \text{ \AA}$ ) or Zn ( $0.82 \text{ \AA}$ ) in the spinel lattice. Snek feels, however, that Cd or Ca ions hold tetrahedric places in the spinel lattice (A).

Paulus, Guillaud etc. [3]... [7] introduce Ca ions up to a concentration of 0.1% mol into the Mn-Zn ferrite. They noticed that calcium addition determines a substantial increase of the  $\mu Q$  product, between initial permeability  $\mu$  and the quality factor  $Q$ , without any increase of the losses by hysteresis or by draw. The results obtained are determined by calcium segregation, introduced in the Mn-Zn ferrite at the crystallites boundaries, a product obtained after the slow cooling of prepared ferrites.

Hrynkiez, Mazanek, Wyderko etc. [8], [9] prepared calcium magnetites, i.e. the  $\text{Ca}_x\text{Fe}_{3-x}\text{O}_4$  system, called calciomagnetite, which is a pure spinel compound for  $x \leq 0.28$ . In the case of fast cooling they noticed a considerable decrease of the real magnetic field and of the Curie temperature as the  $x$  calcium concentration rose. De Sitter, De Grave etc. [10], [11] have carried out measurements of spontaneous magnetization and Mössbauer measurements on the  $\text{Ca}_x\text{Fe}_{3-x}\text{O}_4$  system. They determined the solubility range of calcium in magnetite for the obtention of pure monophasic spinel calciomagnetite and show that all  $\text{Ca}^{2+}$  ions hold tetrahedric places.

The  $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$  system obtained by us was studied as concerns structure [12]... [14], electric properties [15]... [19] and part of the magnetic properties [20].

It is necessary to carry on more thorough investigations on some magnetic properties, viz., initial permeability and Curie temperature and to determine their variation with the calcium content.

To pick up experimental data, calcium was added to Ni-Fe ferrite in two situations:



a) under stoichiometry conditions, which yielded the  $\text{Ca}_x \text{Ni}_{1-x} \text{Fe}_2 \text{O}_4$  system, depending on the  $x$  calcium content, sinterization temperature and cooling way (slow or fast), the compound may be purely spinel monophase or pluriphase [13] ... [19];

b) beyond stoichiometry conditions, when the result is always a pluriphase compound.

The measurements of initial permeability — whose variation with temperature permitted to determine the Curie temperature — were carried out on torus-shaped samples, by using relation

$$(1) \quad \mu = \frac{0.217 \cdot 10^7}{n^2 h \lg(d_e/d_i)} L,$$

where  $n$  is the total number of coils in a single layer, wound on tori and having in the case of all performed measurements the value  $n = 90$  coils;  $d_e$  — the outer diameter of tori, with a value of 26.5 mm;  $d_i$  — the inner diameter of the torus-shaped samples, with a value of 14.5 mm;  $h$  — the height of samples equal to 6 mm;  $L$  — the inductivity in mH, measured with the help of an inductivity bridge.

The experimental device used to measure inductivity is shown in Fig. 1. It is made up a Tesla set, with a Maxwell bridge for low inductivity, TM-382 type, a generator for fixed frequencies TM-512 type and a null detector TM-672 type.

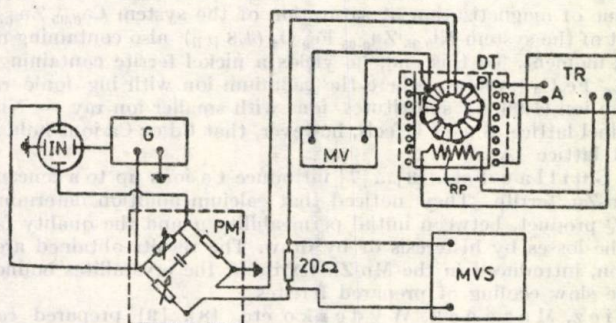


Fig. 1 — Schematic diagram of the installation for the measurement of initial permeability: MB—Maxwell bridge; G — generator of fixed frequencies; NI — null indicator; TD — thermocouple; MV—millivoltmeter, SMV — selective millivoltmeter; PR — platinum thermoresistance; TR — thermoregulator.

The sample winding is connected in series to a purely ohmic resistance of 20  $\Omega$ . With the help of a selective millivoltmeter one can measure the voltage drop, namely the value of the current passing through the winding.

Based on relation

$$(2) \quad H = \frac{nI}{l} = \frac{nI}{2\pi r},$$



where  $l = 2\pi r$  while

$$(3) \quad r = \frac{r_e - r_i}{\ln(r_e/r_i)},$$

one can determine the value of the magnetic field.

All measurements were performed for the same value of the magnetic measurement field, viz. 5 mOe and for the same frequency (1 000 Hz).

Since ferrite samples used in the works [12], [14]... [20] were prepared without presinterization processes and sinterization was performed at temperatures ranging between 1 175 and 1 250°C, considered as insufficient for a full sinterization process, new samples were obtained. These were presinterized for one hour at a temperature of 1 000°C and then sinterized for 6 hours at 1 300°C or for 4 hours at 1 350°C.

Sinterized samples were either fast-cooled, by water quenching at a rate of 800°C/min, or slow-cooled, with the oven, at a rate of 1.5°C/min.

Figs. 2 and 3 show the variation of initial permeability with temperature for some samples of the  $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$  system, sinterized at different temperatures and cooled slowly or fast, where Ca was introduced

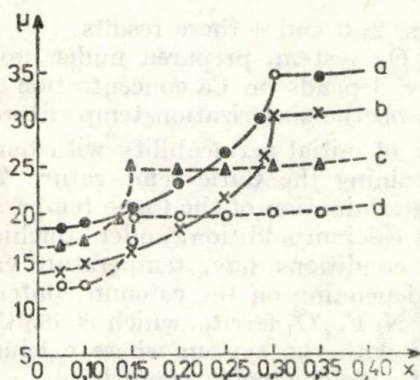


Fig. 2 — Variation of initial permeability with temperature in  $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$  ferrites for different values of the calcium concentration  $x$ : a) ferrite  $\text{Ca}_{0.15}\text{Ni}_{0.85}\text{Fe}_2\text{O}_4$  sinterized at 1 300°C, slowly-cooled; b) ferrite  $\text{Ca}_{0.2}\text{Ni}_{0.8}\text{Fe}_2\text{O}_4$  sinterized at 1 350°C, fast-cooled; c) the system  $\text{NiFe}_2\text{O}_4 + 0.1\%$  mol Ca sinterized at 1 350°C and slowly-cooled.

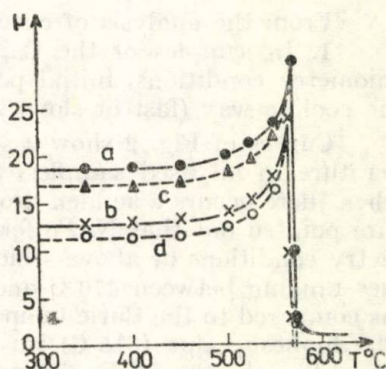


Fig. 3 — Variation of initial permeability with temperature for the ferrites  $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$  with the same calcium concentration  $x$  but differently prepared: a) ferrite  $\text{Ca}_{0.15}\text{Ni}_{0.85}\text{Fe}_2\text{O}_4$  sinterized at 1 350°C and fast-cooled; b) ferrite  $\text{Ca}_{0.15}\text{Ni}_{0.85}\text{Fe}_2\text{O}_4$  sinterized at 1 350°C and fast-cooled; c) ferrite  $\text{Ca}_{0.15}\text{Ni}_{0.85}\text{Fe}_2\text{O}_4$  sinterized at 1 350°C and slowly-cooled; d) ferrite  $\text{Ca}_{0.15}\text{Ni}_{0.85}\text{Fe}_2\text{O}_4$  sinterized at 1 300°C and slowly-cooled.

under or beyond stoichiometry conditions. Fig. 4 reveals the variation of initial permeability with calcium concentration  $x$  in the ferrites of system  $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$  sinterized at different temperatures and cooled fast or slowly.

The Curie temperature was determined from variation curves of the initial permeability with temperature. Its value was obtained by extra-



polating the lower inflexion point of the curves, according to Fig. 2. Fig. 5 shows the variation of the Curie temperature with the Ca concentration  $x$  in the samples of the  $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$  system, fast or slowly cooled, where calcium was introduced in the ferrite under stoichiometry conditions.

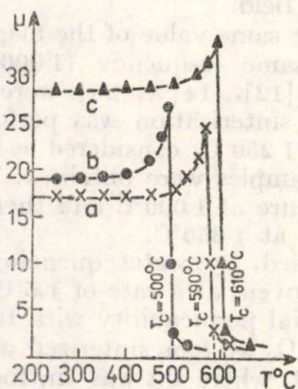


Fig. 4 — Variation of initial permeability with calcium concentration  $x$  of ferrites  $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$ : a) ferrite  $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$  sinterized at 1350°C and cooled fast; b) ferrite  $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$  sinterized at 1300°C and cooled fast; c) ferrite  $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$  sinterized at 1350°C and cooled slowly; d) ferrite  $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$  sinterized at 1300°C and cooled slowly.

From the analysis of curves in Figs. 2, 3 and 4 there results:

1. In samples of the  $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$  system, prepared under stoichiometry conditions, initial permeability depends on Ca concentration  $x$ , the cooling way (fast or slow) as well as on the sinterization temperature.

Curves of Fig. 2 show a steady rise of initial permeability with temperature in all used samples, up to attaining the Curie temperature  $T_c$  when there occurs a sudden drop. The determination of the Curie temperature pointed out that Ni-Fe ferrites with calcium addition, under stoichiometry conditions or above stoichiometry conditions, have temperature values ranging between 470°C and 615°C, depending on the calcium content (as compared to the Curie temperature of  $\text{NiFe}_2\text{O}_4$  ferrite, which is 585°C). The highest value (615°C) was obtained with the system where calcium was added to the Ni-Fe ferrite above the stoichiometry condition.

For the same stoichiometric composition of ferrite, initial permeability increases with the rise of the sinterization temperature as crystallite dimensions increase in the same direction.

2. For samples with the same stoichiometric composition, sinterized at the same temperature, initial permeability depends on the cooling way — fast or slow. It attains higher values with fast-cooled samples where calcium solubility in the lattice is more pronounced than in slow-cooled ferrites, with a more obvious segregation process.

It is seen that all samples with the same stoichiometric composition, sinterized at 1300°C or 1350°C and cooled fast or slowly, have approximately the same value of the Curie temperature, which points out that these ferrites are fully formed from a chemical stand-point and, hence, the Curie temperature is constant.

Curves of Fig. 4 show that initial permeability increases very little with the increase of  $x$ , for small concentrations ( $x < 0.05$ ), when the process of Ca segregation at the crystallite boundaries is prevailing.



For higher calcium concentrations, viz.  $0.05 < x < 0.15$ , in slowly cooled ferrites, and  $0.05 < x < 0.30$ , in fast-cooled ferrites — permeability increases somewhat proportional to the calcium content, because of calcium solubility in the lattice. When reaching the peak of calcium solubility in the lattice ( $x = 0.15$  for slowly-cooled ferrite and  $x = 0.30$  for fast-cooled ferrite), there occurs a new stage (Fe-Ca), as revealed by a sharp increase of initial permeability.

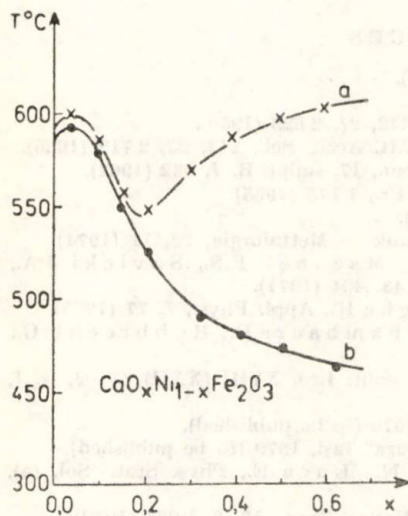


Fig. 5 — Dependence of Curie temperature on calcium concentration  $x$  in ferrites  $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$ : a) ferrite  $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$  sinterized at  $1350^\circ\text{C}$  and cooled slowly; b) ferrite  $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$  sinterized at  $1350^\circ\text{C}$  and cooled fast.

For higher concentrations than those corresponding to the maximum solubility limit, as new stages occur, the process of calcium segregation at the crystallite boundaries is intensified and initial permeability is maintained approximately constant, exhibiting a non-significant monotone increase.

Fig. 5 shows the variation curve of the Curie temperature with calcium concentration  $x$  in the  $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$  system, both for fast-cooled samples and for slowly-cooled ones which are sinterized at the same temperature.

It should be noticed that for small concentrations ( $x < 0.05$ ), the Curie temperature increases with  $x$ , more obviously in slowly-cooled samples than in fast-cooled ones, the value of the Curie temperature being higher than that of the simple ferrite  $\text{NiFe}_2\text{O}_4$ .

There follows a sudden drop of the Curie temperature, which occurs in the range  $0.05 < x < 0.15$  for slowly-cooled ferrites, and  $0.05 < x < 0.30$  for fast-cooled ones. Above these values ( $x > 0.15$  for slowly-cooled ferrites, and  $x > 0.30$  for fast-cooled ones), the variation of the Curie temperature is modified, namely the Curie temperature is rising in slowly-cooled ferrites as  $x$  increases, while in fast-cooled ones the Curie temperature drops steadily, but much more slowly.

Calcium concentrations  $x$  corresponding to the inflexion points in variation curves  $T_c = f(x)$ , viz. values  $x \simeq 0.15$  for slowly-cooled ferrites, and  $x \simeq 0.30$  for fast-cooled ones, point to maximum solubility limits of calcium in the spinel lattice, and to the appearance of new stages corresponding to ferrite Ca-Fe, above these values of  $x$ .

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## PERMEABILITATEA ÎNȚĂLĂ ȘI TEMPERATURA CURIE LA COMPUȘII SISTEMULUI $\text{Ca}_x \text{Ni}_{1-x} \text{Fe}_2 \text{O}_4$

(Rezumat)

Introducerea CaO în ferita  $\text{NiFe}_2 \text{O}_4$  determină, funcție de concentrarea  $x$  în ioni de calciu, de temperatura de sinterizare și de modul de răcire (rapid sau lent), formarea unui compus spinelic monofazic ( $\text{Ca}_x \text{Ni}_{1-x} \text{Fe}_2 \text{O}_4$ ) sau a unui compus plurifazic.

Din variația permeabilității inițiale funcție de temperatura de sinterizare, de modul de răcire și de concentrația în ioni de calciu, cit și din variația temperaturii Curie a compuşilor formați, funcție de concentrația în ioni de calciu, atît la feritele răcite lent cit și la cele răcite rapid, s-a putut determina gradul de solubilitate al calciului în ferita  $\text{NiFe}_2 \text{O}_4$ , adică s-au putut determina atît concentrațiile în ioni de calciu pînă la care se formează ferita spinelică monofazică  $\text{Ca}_x \text{Ni}_{1-x} \text{Fe}_2 \text{O}_4$  cit și concentrațiile deasupra cărora compusul obținut este plurifazic.



## STUDY OF THERMALLY STIMULATED DISCHARGE IN THERMOELECTRETS FROM POLYACRYLONITRILE

BY

EUGEN NEAGU, M. LEANCĂ and RODICA NEAGU

**1. Introduction.** Dielectric materials have the property of polarizing under the action of an external electric field. If this field is applied simultaneously with the sample heating, and the sample is cooled up also under the action of the electric field, we notice that the sample has a remanent electric charge, whose magnitude is characterized by superficial density of charge  $\sigma$ . An thermoelectret is thus obtained, which represents a piece which produces an external electric field. Though the first electret was obtained in 1920 by Eguchi [1], their study was greatly intensified after 1962, when Sessler and West [2] have achieved a microphone with electret. Since then, the number of applications of devices based on electrets has greatly increased. One can quote: infrasonic microphones with electret [3] with low cutoff frequencies of  $10^{-3}$  Hz, ultrasonic microphones with electret [4], [5], gas analyser using a microphone with electret from plastic sheet [6], [7].

The properties of the electret depend both on the chemical structure of the used material and on the morphological structure, which may vary within a wide range in polymers, depending on the formation condition of the polymer.

The charge of the electret can originate from: the orientation of the polar groups under the action of the electric field, the shifting of ions and electrons originating from thermally activated processes, the charge accumulated at the metal-dielectric contact. The charge has a sign opposite to the formation electrode and is called a heterocharge. Since the contact between the electrode and the electret is not intimate, in the remaining interstice of air there occurs a dielectric breakdown of the air. This results in the depositing of positive ions on one side of the electret, and of electrons on the other side in a much lower number. This charge is called a homocharge.

For the quantitative study of the electret effect the method of the thermally stimulated current (TSC) has been used of late.

The non isothermal discharge was first used in 1936 [8] as a method of research. The theoretical foundations of the method were laid in 1964, when Bucci and Fieschi [9] developed the theory of the method of the ionic thermocurrent (ITC).

The method was initially used for the thermally activated release of dielectric polarization in crystals of micromolecular substance [10], [11]. The method was subsequently applied in the study of electrets [12] ... [16].

The present work provides first a general and complete survey of the method of thermally stimulated discharge, which is then applied to the study of electrets prepared from polyacrylonitrile (PAN) films.

A method for determination of the charge resulting from dipole alignment  $Q_p$  is proposed and it is found that this charge has a small  $p$  contribution to the total charge  $Q_T$ .

**2. Theory.** Bucci and Fieschi's theory refers to polar dielectrics and permits to make a complete description of the dependence of thermoionic currents on temperature. From the study of these currents one can determine the characteristic parameters of dipoles  $\tau_0$  and  $W$  from a single measurement. These parameters can be determined also by dielectric



measurements. The determination of these parameters by this latter way is very different for electrets.

Without discussing details, the method consists in the measurement of the depolarization (or discharge) current of an electret, depending on temperature.

At the moment  $t_0$  (Fig. 1), an electric field  $E$  is applied to the sample. The sample has the polarization temperature  $T_p$ , which is maintained constant up to the moment  $t_1$ . The time interval  $t_1 - t_0$  should be much longer than the relaxation times of dipoles, so that one could

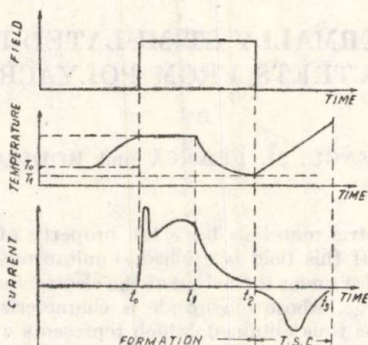


Fig. 1 — Sample preparation and thermally stimulated discharge process.

consider that a state of equilibrium for the respective temperature of the oriented dipoles was reached. From this moment, the sample is cooled and the electric field is still applied up to temperature  $T_1$ . At the moment  $t_2$ , when the sample reached the expected temperature  $T_1$ , the exterior electric field is removed. At this moment, the electret formation is finished. Homocharges cannot appear when metallic electrodes have been deposited on the surface of the dielectric from which the electret was obtained, for instance by vacuum metallization.

The superficial charge density can be determined by one of the known methods. From the study of the time variation of the superficial charge density, conclusions can be drawn concerning the life time of the electret and some data on the nature of the electret charge.

The TSD method permits to determine the relaxation time, the activation energy and the total charge of the electret. The disadvantage of this method is that the electret is destroyed by heating.

In the hypothesis that the charge mainly originates from orientation of dipoles, the expression of density of discharge currents can be written

$$(1) \quad J(T) = \frac{Np^2 \alpha E_p}{k T_p} \left[ \tau_0 \exp \left( \frac{W}{kT} \right) \right]^{-1} \exp \left( - \int_0^T \left[ b \tau_0 \exp \left( \frac{W}{kT'} \right) \right]^{-1} dT' \right),$$

where  $N$  — number of dipoles from the volume unit;  $p$  — dipole moment;  $\alpha$  — polarizability (for freely rotating dipoles  $\alpha = 1/3$ );  $E_p$  — intensity of the polarization field;  $k$  — Boltzmann's constant;  $T_p$  — polarization temperature;  $\tau_0$  — a constant;  $W$  — activation energy of relaxation;  $b$  — heating rate ( $T = T_0 + bt$ ).

The total charge of the electret is given by

$$(2) \quad Q_r = \int_0^\infty i(t) dt.$$

If in (2) one replaces (1), one obtains a value which is not dependent on the sample thickness but depends on  $E_p$ ,  $T_p$ .



Differentiating expression (1) and setting condition

$$(3) \quad \frac{dJ}{dT} = 0$$

one obtains the temperature for which the current is maximum

$$(4) \quad T_m^2 = \frac{\tau_0 b W \exp\left(\frac{W}{kT_m}\right)}{k}.$$

Thus, if the charge is determined by the polarization of the dielectric, the temperature where the current is the highest is independent from  $E_p$  and  $T_p$ . This way one can check whether the charge is determined by a uniform orientation of the dipoles, namely by the movement of the charge on microscopic distances, followed by their capture in traps or by a space charge. From relation (4), one can determine  $\tau_0$ .

For not too high temperatures, Eq. (1) can be written as

$$(5) \quad \ln J(T) = C \exp\left(-\frac{W}{kT}\right),$$

or

$$(6) \quad \ln J(T) = C - \frac{W}{kT},$$

where  $C$  is a constant in the field of low temperatures. If  $\ln J(T)$  is plotted as function on  $1/T$ , there appears a straight line.

By its slope one can determine the activation energy  $W$ . This is the method of G a r l i c h and G i b s o n.

The number of dipoles contained in the volume unit of the respective dielectric can be appreciated since from Langevin's function one gets

$$(7) \quad P_0 = \frac{N \alpha p^2 E_p}{k T_p},$$

where  $P_0$  is the initial polarization.

On the other hand

$$(8) \quad P_0 = \int_0^\infty \tau(t) dt,$$

and, therefore, from (7) and (8) one can determine  $N$ .

$W$  and  $\tau_0$  can be obtained not only from the initial portion of the curve but also using the whole curve  $i(T)$ .

Since

$$(9) \quad \tau(T) = -\frac{P(T)}{J(T)} = \frac{1}{J(T)} \int_{t(T)}^\infty J(t') dt',$$

where  $J(T)$  is the current density for temperature  $T$ , and

$$(10) \quad \tau(T) = \tau_0 \exp\left(\frac{W}{kT}\right),$$

from (9) and (10) there results

$$(11) \quad \ln \tau(T) = \ln \tau_0 + \frac{W}{kT} = \ln \left[ \frac{1}{J(T)} \int_{t(T)}^\infty J(t') dt' \right].$$

If relation (11) is plotted by a straight line, one can conclude that it is a uniform polarization process. The slope of the straight line gives  $W$  and the intercept of this line gives  $\tau_0$ . Cowell and Woods' method [11] can be used to the same effect.

The presented theory is applied when dipoles are characterized by a single relaxation time, this being an idealized case. Actually, the activation energy and the relaxation times are distributed in an interval round an average value. The equation of the thermally stimulated current for the case where the distribution function of the dipole activation energy is expressed by  $G(W)$  is given by Hino [21]

$$(12) \quad J(T) = \int_0^{\infty} \left\{ G(W) \frac{p^2 w E p_{\infty}}{k T_p \tau_0(w)} \right\} \exp \left\{ -\frac{1}{\tau_0(w) b} \int_{T_0}^T \exp \left( -\frac{W}{k T'} \right) dT' \right\} \exp \left( -\frac{W}{k T} \right) dW,$$

where  $p(W)$  and  $\tau(W)$  depend on  $W$ ,  $G(W)dW$  is the number of dipoles from the volume unit, whose activation energy ranges between  $W$  and  $W + dW$ . In establishing relation (12), it is considered that dipoles do not interact.  $G(W)$  can be determined from (12).

As shown,  $W$  and  $\tau_0$  can be determined also by dielectric measurements, namely by studying the variation of the dielectric constant and of the tangent of the loss angle with frequency and temperature. It is difficult to measure these magnitudes, both in high frequencies and in low frequencies ( $\approx 0,1$  Hz).

From TSD data, one can determine not only  $\tau_0$  and  $W$ , but also  $\epsilon'$  and  $\epsilon''$  since Debye's equations can be written

$$(13) \quad \epsilon' - \epsilon_{\infty} = \frac{Q_T}{1 + \omega^2 \tau^2} \frac{1}{C_0 V},$$

$$(14) \quad \epsilon'' = \frac{Q_T \omega \tau}{1 + \omega^2 \tau^2} \frac{1}{C_0 V},$$

where  $\epsilon_{\infty}$  is the dielectric constant for high frequencies;  $Q_T$  — total charge obtained from the TSD curve;  $C_0$  — capacity calculated by replacing the sample with vacuum;  $V$  — applied voltage;  $\omega = 2\pi\nu$ ,  $\nu$  — frequency.

**3. Experimental Procedure.** The investigated PAN samples were obtained in a film form, by the depositing of the solution polymer, with a  $3.5 \times 10^{-2}$  mm thickness and 12 mm diameter. To remove solvent traces and water from the samples, these were maintained in a vacuum ( $10^{-4}$  torr), at a temperature of  $50^\circ\text{C}$  for 4 days.

In order to achieve a fine electric contact and to avoid the formation of the homocharge, an Ag layer was deposited on the surface of samples by vacuum evaporation. The scheme of the experimental installation is shown in Fig. 2. Samples were kept for 2 hours at the polarization temperature and then were slowly cooled down, in the presence of the electric field, up to room temperature.

The current was measured with a vibrating reed electrometer and was recorded on the  $Y$  axis of a  $X-Y$  recorder. Temperature was measured by a thermocouple placed in the vicinity of the sample. The signals of the thermocouple were recorded on the  $X$  axis of the recorder.

Measurements were made for linear increasing temperature.

**4. Results.** Fig. 3 shows TSD spectra for electrets made of PAN. The polarization temperature was of 413 K for all samples and the intensity of the electric polarization field ranged between 4.7 and 7.10 kV/cm.



It is noticed that in all the studied cases, spectra show a single peak, round the temperature of 371 K.

The calculation of the surface based on relation (2) permitted to determine the total charge  $Q_T$  of electrets for the three cases under study. Fig. 4 shows the dependence of the total charge  $Q_T$  on the value of the intensity of the electric field  $E_p$ . It is noticed that there is a linear relation between  $Q_T$  and  $E_p$  within the range of the electric fields used.

To characterize the electret state in the respective materials the activation energy of relaxation  $W$  and the frequency factor  $\tau_0$  were deter-

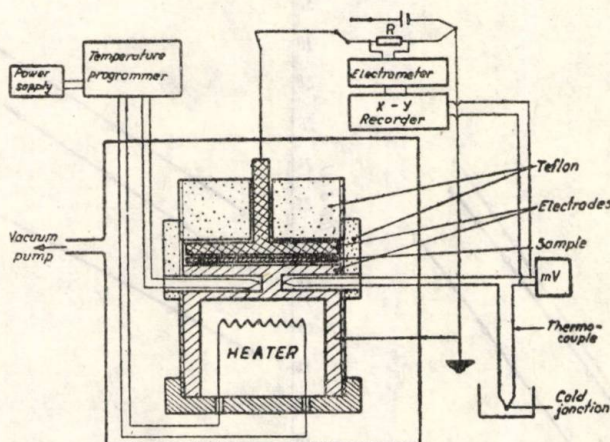


Fig. 2 — Schematic diagram of the apparatus for measuring TSD current.

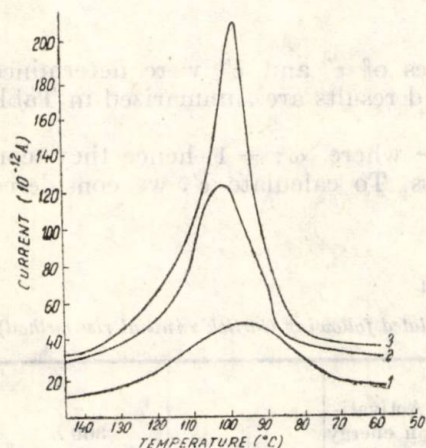


Fig. 3 — TSD current spectra obtained from polyacrylonitrile film (polarizing temperature = 413 K, polarizing time 2 hr); the curves 1, 2 and 3 correspond to  $E_p = 4.7, 7.4$  and  $7.1$  kV/cm respectively.

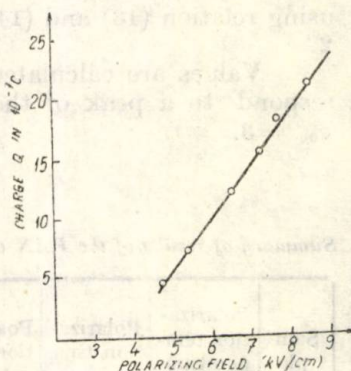


Fig. 4 — Charge released to external circuit vs. polarizing field.



mined. A first determination was based on relations (6) and (4), using the initial portion of curves from Fig. 3, and results are plotted in the graphs of Fig. 5, where  $\lg I$  vs  $1000/T$  was plotted. Based on (11), to this effect were also used the graphs obtained by the semilogarithmic plotting of  $\tau$  as a function of  $1/T$ , which are presented in Fig. 6. Magnitudes determined from the analysis of graphs in Figs. 5 and 6 are given in Table 1.

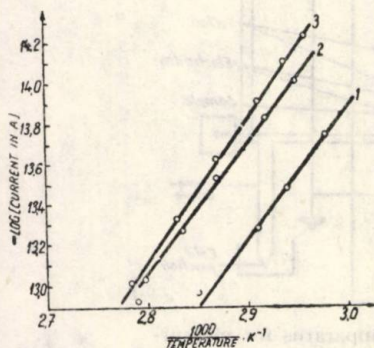


Fig. 5 — Arrhenius plots obtained from the initial rise of the TSD peak.

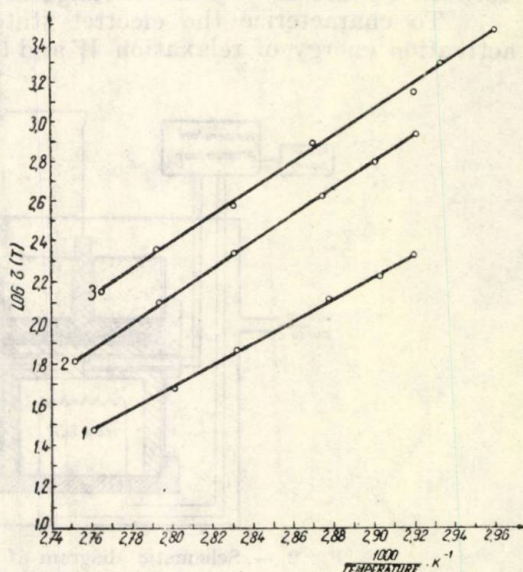


Fig. 6—Log  $\tau$  vs.  $1/T$  curves obtained from Eq. (11).

From TSD curves obtained, values of  $\epsilon'$  and  $\epsilon''$  were determined using relation (13) and (14). The obtained results are summarized in Table 2.

Values are calculated for the case where  $\omega\tau = 1$ , hence they correspond to a peak of the dielectric loss. To calculate  $\epsilon'$ , we considered  $\epsilon_{00} = 3$ .

Table 1

Summary of results of the PAN electrets (data calculated following Garlick's initial rise method)

| Sam-<br>ple | Polariza-<br>tion tem-<br>perature<br>K | Polariza-<br>tion time<br>hr | Polaiza-<br>tion field<br>V/cm | Tempera-<br>ture cor-<br>respond-<br>ing to<br>peak<br>K | Activati-<br>on energy<br>eV | $\tau_0$<br>sec       | $\tau$<br>300 K<br>sec |
|-------------|-----------------------------------------|------------------------------|--------------------------------|----------------------------------------------------------|------------------------------|-----------------------|------------------------|
| 1           | 413                                     | 2                            | 4 700                          | 370                                                      | 1,26                         | $2.05 \cdot 10^{-15}$ | $3.48 \cdot 10^{-6}$   |
| 2           | 413                                     | 2                            | 5 400                          | 373                                                      | 1,24                         | $6.32 \cdot 10^{-15}$ | $4.18 \cdot 10^{-6}$   |
| 3           | 413                                     | 2                            | 7 100                          | 371                                                      | 1,37                         | $5.6 \cdot 10^{-17}$  | $6.77 \cdot 10^{-6}$   |



Table 2

Summary of results of the PAN electrets (data calculated following whole curve method)

| Sam-<br>ple | Polariza-<br>tion tem-<br>perature<br>K | Polariza-<br>tion time<br>hr | Polariza-<br>tion field<br>V/cm | Tempera-<br>ture co-<br>respon-<br>ding to<br>peak K | Activati-<br>on energy<br>eV | $\tau_0$<br>sec       | $\tau$<br>300 K<br>sec |
|-------------|-----------------------------------------|------------------------------|---------------------------------|------------------------------------------------------|------------------------------|-----------------------|------------------------|
| 1           | 413                                     | 2                            | 4 700                           | 370                                                  | 1.06                         | $4.06 \cdot 10^{-14}$ | $2.9 \cdot 10^4$       |
| 2           | 413                                     | 2                            | 5 400                           | 373                                                  | 1.28                         | $1.01 \cdot 10^{-16}$ | $3.9 \cdot 10^5$       |
| 3           | 413                                     | 2                            | 7 100                           | 371                                                  | 1.38                         | $5.14 \cdot 10^{-18}$ | $1.02 \cdot 10^6$      |

$\epsilon'$  was determined experimentally using a bridge for small capacities and the value 18 was obtained on an average for studied samples.

5. Discussions and Conclusions. 1. Parameters  $W$ ,  $\tau_0$  and  $\tau$  were calculated on the basis of experimental data. Data obtained by the two methods (Tables 1 and 2) are close. It is noticed, however, that values calculated using only the initial portion of the curve than those obtained using the whole TSD curves are closer. Consequently, it may be concluded that  $W$  should be determined using the whole TSD curve. The choice of the method of determination of the activation energy is important as its small variations determine important variations of  $\tau_0$  and  $\tau$ .

2. The total charge of the electret increases linearly with the increase in the intensity of the polarizing field (Fig. 4). It is noticed that one cannot establish a linear correlation between the total charge of the electret and its life time. While the total charge of sample 3 is 4 times as high as the charge of sample 1, its relaxation time at room temperature is about twice higher.

3. A method of quantitative determination of the contribution of the polarization charge to the total charge of the electret is suggested. It is noticed that values of  $\epsilon'$  and  $\epsilon''$  (Table 3) are higher than those obtained by bridge measurements. These differences can be explained if account is taken of the fact that the experimentally determined charge does not contain only the charge originating from dipole orientation. The contribution of the polarization charge to the total charge can be determined as follows. Knowing the values of  $\epsilon'$  and  $\epsilon''$  and  $\epsilon_\infty$ , from relations (13)

Table 3

Values of dielectric parameters and of the charge of PAN electrets

| Sam-<br>ple | $\epsilon_{00}$ | $\epsilon'$<br>measu-<br>red | $\epsilon'$<br>calculated | $\epsilon''$<br>calculated | Charge<br>$Q_T$<br>As | Charge<br>$Q_T$<br>As |
|-------------|-----------------|------------------------------|---------------------------|----------------------------|-----------------------|-----------------------|
| 1           | 3               | 18                           | $3.92 \cdot 10^2$         | $3.89 \cdot 10^2$          | $4.48 \cdot 10^{-7}$  | $1.72 \cdot 10^{-8}$  |
| 2           | 3               | 18                           | $5.58 \cdot 10^2$         | $5.55 \cdot 10^2$          | $7.38 \cdot 10^{-7}$  | $2 \cdot 10^{-8}$     |
| 3           | 3               | 18                           | $9.53 \cdot 10^2$         | $9.40 \cdot 10^2$          | $1.8 \cdot 10^{-8}$   | $2.82 \cdot 10^{-8}$  |



and (14), one can determine the value of the polarization charge and the difference from charge  $Q_T$ , experimentally determined from TSD curves represents the contribution of the other types of charge. Performed calculations pointed out that the polarization charge might represent  $3.80 \times 10^{-2}$  from the total charge. Following the pathway suggested by Perlman [16], we have calculated the number of dipoles per monomer necessary for the electret to have a charge equal to  $Q_T$  resulting from dipole orientation. This calculation has led to a number of 70 dipoles per monomer, which confirms that the charge determined by the orientation of permanent dipoles has a smaller contribution to  $Q_T$ .

4. Though the contribution of the charge resulting from the orientation of permanent dipoles is very small, the total charge  $Q_T$  appears like a polarization charge and the main contribution is given by the electrons and ions moved on microscopic distances and captured in traps. To a smaller extent, to  $Q_T$  there also contributes the charge accumulated at the metal-dielectric contact, or the charge injected from the electrodes.

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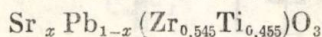
#### STUDIUL DESCĂRCĂRII STIMULATE TERMIC ÎN TERMOELECTREȚII DIN POLIACRILONITRIL

(Rezumat)

Termoelectreții din poliacrilonitril obținuți pentru diferite intensități ale câmpului electric, au fost studiați prin metoda descărcării stimulate termic. Măsurătorile au fost făcute în vid, pe probe încălzite cu viteză constantă.

S-a observat că energia de activare și timpul de relaxare depind de valoarea câmpului de polarizare. Sarcina este uniform distribuită în tot volumul probei. Este propusă o metodă de determinare a sarcinii care rezultă din orientarea dipolilor  $Q_p$  și se găsește că această sarcină aduce o contribuție mică la sarcina totală  $Q_T$ . Rezultă că sarcina totală este produsă în principal din deplasarea electronilor și ionilor pe distanțe microscopice și din captarea acestora.



RECHERCHES CONCERNANT LA PRÉPARATION ET LES  
PROPRIÉTÉS DU COMPOSÉ PIÉZOCÉRAMIQUE

PAR

IRINA JEMNA et C. IOAN

**1. Introduction.** Afin d'obtenir des matériaux piézo céramiques dont les propriétés concordent avec les qualités nécessaires aux filtres pour les fréquences intermédiaires on a du considérer les composés à base de zirco-titanate de plomb avec addition de Sr et spécialement le composé céramique  $\text{Sr}_{0,05} \text{Pb}_{0,95} (\text{Zr}_{0,545} \text{Ti}_{0,455}) \text{O}_3$  [1] ... [5].

La technologie de préparation est celle utilisée habituellement dans la technique céramique [2], [5], [6]. On utilise donc l'homogénéisation humide, une calcination, durant environ 2 heures, à la température de 900°C et le frittage, durant 2 heures, à la température de 1250°C. Pour assurer la stoechiométrie du composé le frittage doit être exécuté en atmosphère d'oxyde de plomb. Pour la réaliser on a introduit les échantillons dans des capsules en nickel dont le plancher a été couvert d'une couche en zirconate de plomb grossier.

Les mesures des densités, effectuées par la méthode du pycnomètre, ont conduit, pour les échantillons ainsi préparés, aux valeurs d'environ 5 g/cm<sup>3</sup>, par rapport à 7,9 g/cm<sup>3</sup> — la valeur de la densité de rayons X.

Les échantillons ont été polarisés en air, par l'application d'un champ électrique continu de 5 kV/cm, pendant 15 minutes, la température étant de 80°C. La température a été diminuée jusqu'à 30°C tandis que l'intensité du champ a été maintenue constante.

L'intervalle entre le traitement de polarisation et la détermination des paramètres diélectriques et piézoélectriques a été de 24 heures.

**2. Propriétés diélectriques** (permittivité et pertes diélectriques). La mesure de la permittivité et des pertes diélectriques, exprimées par la tangente de l'angle de pertes ( $\text{tg } \delta$ ), a été effectuée à l'aide d'un pont pour admittances type SWM 3—2, conformément au schéma-bloc représenté dans la fig. 1, où: 1 — oscillateur TR-0202 Orion, type RC en décades, avec une plage de fréquence 10 Hz — 1 MHz et la stabilité de  $10^{-5}$ ; 2 — fréquencemètre PFL-20, pour contrôler la fréquence; 3 — pont de mesure SWM 3—2, mesurant l'admittance dans le domaine 30 Hz — 1,5 MHz; 4 — indicateur de nul — millivoltmètre sélectif type TT-1 302, dont la sensibilité est de 20  $\mu\text{V}/\text{div}$  dans la plage 10 kHz — 1 MHz et



respectivement  $2 \mu\text{V/div}$  dans la plage 10 Hz — 10 kHz ; 5 — cellule de mesure, type, TR-9 701, en fait un condensateur à anneau de garde, écrané contre les champs électriques perturbateurs, avec possibilités de réglage et contrôle de la distance entre les armatures. Les erreurs de mesure des capacités n'ont pas dépassées 1%.

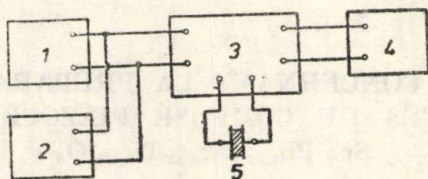


Fig. 1 — Le schéma-bloc du montage de mesure des propriétés diélectriques.

La permittivité et les pertes diélectriques des céramiques piézoélectriques polarisées dépendent de fréquence tant à cause des effets de polarisation interfaciale (provoquée par une conductibilité irrégulière dans le domaine de basses fréquences), de relaxation des dipôles associée au déplacement des impuretés et des parois des domaines, quant à la présence de l'effet piézoélectrique même.

Utilisant des échantillons-disques, en composé piézo-céramique réalisé, de 10 mm diamètre, 10 mm hauteur et 8 mm diamètre des électrodes, la mesure conventionnelle à la fréquence de 1 kHz a conduit à une valeur d'environ 1 500 pour  $\epsilon_r$  et aux valeurs sous la limite de sensibilité du pont de mesure pour  $\text{tg } \delta$ .

Dans les fig. 2 et 3 sont représentées les courbes de variation de  $\epsilon_r$  et de  $\text{tg } \delta$  en fonction de fréquence.

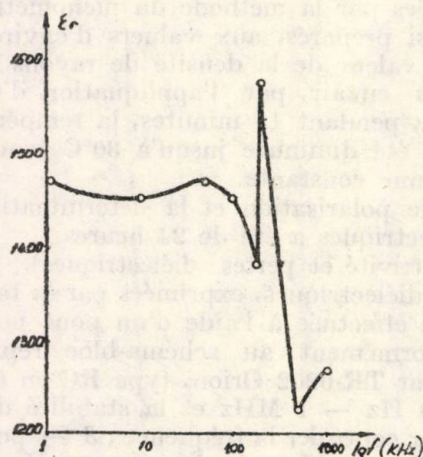


Fig. 2 — Variation de la permittivité diélectrique relative en fonction de la fréquence pour l'échantillon 2.

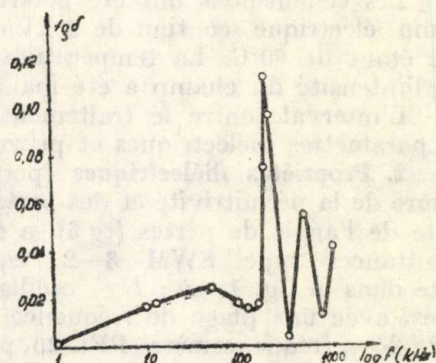


Fig. 3 — Variation des pertes diélectriques en fonction de la fréquence pour l'échantillon 2.



Les maxima présentés dans le voisinage de la fréquence de résonance montrent que les échantillons se comportent comme une piézocéramique courante.

**3. Propriétés piézoélectriques** (fréquence de résonance, d'antirésonance, facteur de couplage piézoélectrique). On calcule le facteur de couplage piézoélectrique en connaissant les valeurs des fréquences de résonance et d'antirésonance. Pour les échantillons disques, polarisés, qu'on a utilisé, il a été nécessaire de déterminer les fréquences de résonance et d'antirésonance radiale.

Le schéma-bloc de l'installation de mesure est représenté dans la fig. 4, où : 1 — oscillateur TR-0202 ; 2 — fréquencemètre type PFL-20 ; 3 — porte-échantillon ; 4 — millivoltmètre TR-1 201 à plage étendue, avec une impédance d'entrée suffisamment élevée pour ne pas perturber le montage de mesure ;  $R_1$ ,  $R_2$  — diviseur qui réalise pratiquement une source de faible résistance interne ;  $R_3$  et  $R_4$  — résistances de mesure

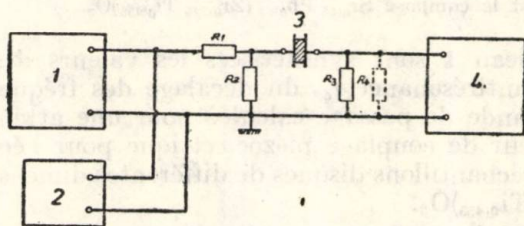


Fig. 4 — Le schéma-bloc du montage de mesure des propriétés piézoélectriques.

(aux bornes desquelles on a mesuré la tension) dont les valeurs ont été choisies en accord avec la valeur de la résistance équivalente du traducteur lorsque la fréquence est celle de résonance, respectivement d'antirésonance. Les résistances  $R_1$ ,  $R_3$  et  $R_4$  sont chimiques, tandis que  $R_2$  est un fil court en manganine. Toutes les connexions ont été exécutées en câble coaxial écrané.

Après avoir déterminé expérimentalement les fréquences de résonance  $f_r$  et d'antirésonance  $f_a$  on a calculé le facteur de couplage piézoélectrique  $k_p$  conformément à l'expression [1], [7], [8]

$$(1) \quad k_p = \sqrt{2,51 \frac{f_a - f_r}{f_r}}.$$

À cause du fait que l'aspect de la courbe de résonance dépend de la géométrie de l'échantillon et des électrodes et de la position des contacts électriques, on a utilisé dans les expériences des échantillons disques de différentes dimensions, comme on voit dans le tableau 1.

Dans les fig. 5, 6, 7 et 8 sont représentées les courbes de résonance pour l'échantillon de référence (Philips 455) et pour les échantillons 1, 2 et 5 du tableau 1, à l'aide desquelles ont été déterminées les fréquences de résonance et d'antirésonance.



Tableau 1

| Nr. crt. | Nature de l'échantillon              | Dia-<br>mètre<br>de<br>l'échan-<br>tillon<br>mm | Rap-<br>port<br>$h/\varnothing$ | Dia-<br>mètre<br>des<br>élec-<br>trodes<br>mm | $f_r$<br>kHz | $f_a$<br>kHz | $f_a - f_r$<br>kHz | $(\Delta f)_{3dB}$<br>kHz | $Q_m$ | $k_p$ |
|----------|--------------------------------------|-------------------------------------------------|---------------------------------|-----------------------------------------------|--------------|--------------|--------------------|---------------------------|-------|-------|
| 1        | Échantillon de référence Philips 455 | 5,6                                             | 0,1                             | 3,6                                           | 455          | 464          | 9                  | 1,6                       | 984   | 0,22  |
| 2        | PZT* 1                               | 10                                              | 1                               | 10                                            | 193          | 195          | 2                  | 1                         | 262   | 0,16  |
| 3        | PZT 2                                | 10                                              | 1                               | 8                                             | 195          | 207          | 12                 | 0,8                       | 279   | 0,4   |
| 4        | PZT 3                                | 10                                              | 0,3                             | 10                                            | 282          | 288          | 6                  | 2                         | 209   | 0,21  |
| 5        | PZT 4                                | 8                                               | 0,1                             | 8                                             | 368          | 378          | 10                 | 4                         | 154   | 0,26  |
| 6        | PZT 5                                | 6,5                                             | 0,1                             | 6,5                                           | 457          | 461          | 4                  | 4                         | 152   | 0,15  |

\* PZT 1 ... 5 est le composé  $Sr_{0,05}Pb_{0,95}(Zr_{0,545}Ti_{0,455})O_3$ .

Dans le tableau 1 sont synthétisées les valeurs des fréquences de résonance  $f_r$  et d'antirésonance  $f_a$ , du décalage des fréquences  $f_r - f_f$ , de la largeur de la bande de passage calculée pour une atténuation de 3 dB  $(\Delta f)_{3dB}$  et du facteur de couplage piézoélectrique pour l'échantillon de référence et pour les échantillons disques de différentes dimensions en composé  $Sr_{0,05}Pb_{0,95}(Zr_{0,545}Ti_{0,455})O_3$ .

4. Conclusions. L'aspect des courbes de résonance et les valeurs des grandeurs décrites dans le tableau 1 permettent de conclure :

1. Les fréquences de résonance  $f_r$  sont comprises entre 192 et 457 kHz, tandis que les fréquences d'antirésonance varient entre 193 et 461

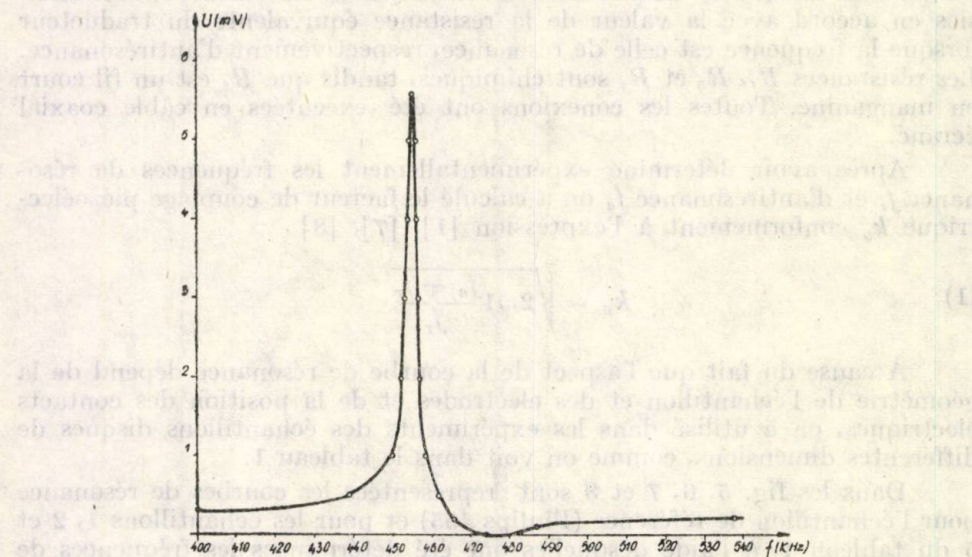


Fig. 5 — La caractéristique tension-fréquence pour l'échantillon de référence.



kHz (par rapport à 455 kHz et respectivement 464 kHz pour l'échantillon de référence).

2. Le décalage des fréquences  $f_a - f_r$  varie entre 2 et 12 kHz tandis que la largeur de la bande de passage à trois décibels atténuation ( $\Delta f$ )<sub>3dB</sub> varie entre 0,8 et 4 kHz (par rapport à 9 kHz et respectivement 1,6 kHz pour l'échantillon de référence).

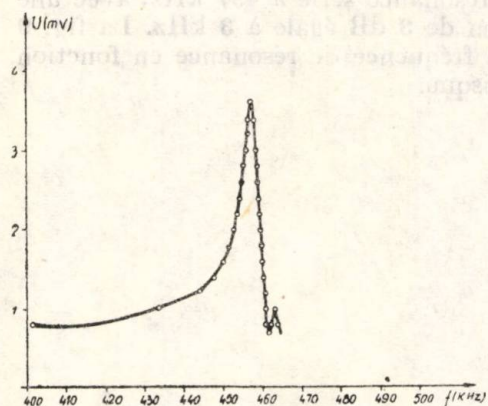


Fig. 6 — La caractéristique tension-fréquence pour l'échantillon 5.

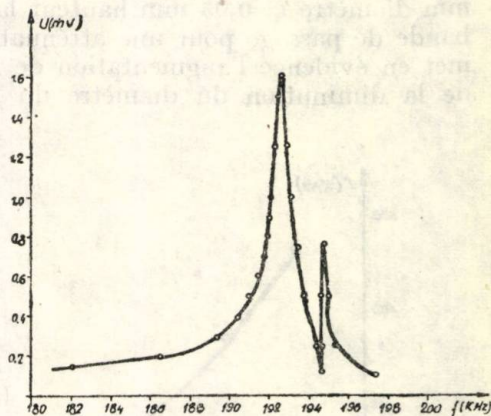


Fig. 7 — La caractéristique tension-fréquence pour l'échantillon 1.

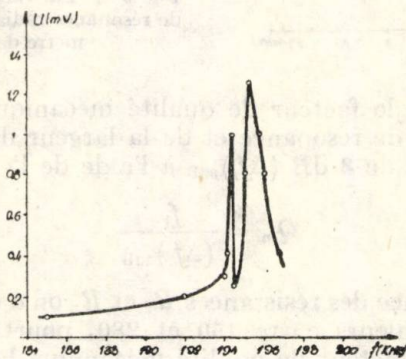


Fig. 8 — La caractéristique tension-fréquence pour l'échantillon 2.

3. Le facteur de couplage piézoélectrique  $k_p$  est compris entre 0,15 et 0,40 (par rapport à 0,22) pour l'échantillon de référence.

4. Pour certains échantillons préparés on met en évidence des fréquences de résonance parasites, c'est à dire on remarque sur les caractéristiques tension—fréquence (fig. 7 et 8) des maxima de moindre amplitude, situés dans le voisinage de la fréquence de résonance. Afin d'éliminer ces maxima parasites par la variation des diamètres des électrodes [2], [9], on a obtenu, en utilisant des diamètres des électrodes de 2 mm moindres



que ceux des échantillons, seulement le changement de la position du maxima parasite, sans arriver à sa élimination complète.

5. Par la variation du rapport hauteur/diamètre ( $h/\varnothing$ ) pour les échantillons disques, de la valeur 1 jusqu'à 0,3 on a obtenu une augmentation d'environ 100 kHz de la fréquence de résonance.

6. La fréquence de résonance augmente aussi quand on diminue le diamètre du disque résonateur [9]; on a obtenu pour l'échantillon de 6,5 mm diamètre et 0,65 mm hauteur la résonance série à 457 kHz, avec une bande de passage pour une atténuation de 3 dB égale à 3 kHz. La fig. 9 met en évidence l'augmentation de la fréquence de résonance en fonction de la diminution du diamètre du disque.

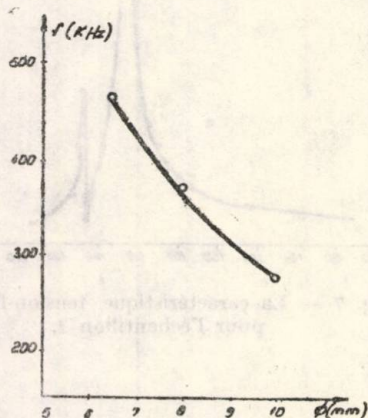


Fig. 9 — La variation de la fréquence de résonance radiale en fonction de diamètre de l'échantillon.

7. On a calculé le facteur de qualité mécanique en utilisant les valeurs de la fréquence de résonance et de la largeur de la bande de passage pour une atténuation de 3 dB ( $\Delta f_r$ )<sub>3dB</sub> à l'aide de l'expression [9]

$$(2) \quad Q_m = \frac{f_r}{(\Delta f_r)_{3dB}}.$$

En éliminant l'influence des résistances  $R_2$  et  $R_3$  on a obtenu pour les échantillons étudiés des valeurs entre 150 et 280, pourtant inférieures à 400, valeur indiquant un facteur de qualité mécanique bon pour les piézocéramiques usuelles [1].

8. Les résultats obtenus, pour ce premier composé  $\text{Sr}_{0,05} \text{Pb}_{0,95} (\text{Zr}_{0,545} \text{Ti}_{0,455}) \text{O}_3$ , pouvant être encadrés, en général, dans les limites acceptables pour les matériaux piézocéramiques usuels, imposent la continuation des recherches par la réalisation d'autres composés, à d'autres éléments d'addition, afin d'obtenir une bonne stabilité thermique, des propriétés diélectriques et piézoélectriques optimales et reproductibles, en utilisant une géométrie des échantillons et des électrodes judicieusement choisie.

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## CERCETĂRI PRIVIND PREPARAREA ȘI PROPRIETĂȚILE COMPUSULUI

PIEZOCERAMIC  $\text{Sr}_x\text{Pb}_{1-x}(\text{Zr}_{0,545}\text{Ti}_{0,455})\text{O}_3$ .

(Rezumat)

Utilizind tehnologia obișnuită de preparare a materialelor piezoceramice s-a realizat compusul  $\text{Sr}_x\text{Pb}_{1-x}(\text{Zr}_{0,545}\text{Ti}_{0,455})\text{O}_3$ , sinterizat timp de 2 ore la temperatura de 1250°C în atmosferă de oxid de plumb.

Proprietățile dielectrice (permitivitate și pierderi dielectrice) corespund piezoceramicele curente. Proprietățile piezoelectrice (frecvență de rezonanță, antirezonanță, factor de cuplaj piezoelectric) au pus în evidență dependența valorilor acestora de geometria probelor și a electrozilor (grosimea probelor, diametrul electrozilor raportat la cel al probelor, raportul grosime-diametru), obținind la unele probe, pentru frecvența de rezonanță și factorul de cuplaj piezoelectric, valori comparabile cu cele ale unei probe de referință (Philips 455), utilizată curent în tehnica filtrelor de frecvență intermediară.

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## RECENZII

### BOOK REVIEWS

### РЕЦЕНЗИИ

И. С. АРЖАННЫХ, *Аналитический аппарат конформных отображений*. ВINITИ, Ташкент, 1976.

The monograph contains author's research on the constructive theory of a class of mappings called by him „generalized conformal mappings” (GCM, in russian *расширенные конформные отображения* — PKO). These mappings generalize the conformal ones (CM) to an  $s + 1$ -dimensional space,  $s \geq 1$ , whose elements

$$(1) \quad x = x_0 + \sum_{n=1}^s x_n \varepsilon_n$$

are called „complexes” and form a commutative  $\mathbb{C}$ -algebra generated by the „places”  $\varepsilon_0 = 1, \varepsilon_1, \dots, \varepsilon_s$  with the law of composition (LC):

$$(2) \quad \varepsilon_j \varepsilon_k = \gamma_{jk}^0 + \sum_{n=1}^s \gamma_{jk}^n \varepsilon_n.$$

The space is denoted by  $\Pi_\sigma$  and endowed with a metric form

$$(3) \quad \chi[x] = \prod_{h=0}^p (\tau_h[x]) n_h, \quad \text{where}$$

$$(4) \quad \tau_h[x] = x_0 + \sum_{n=1}^s \alpha_{nh} x_n, \quad n_0 \geq \dots \geq n_p > 0, \quad n_h \in \mathbb{Z}, \quad \sum_{h=0}^p n_h = s + 1, \quad \alpha_{nh} \in \mathbb{C}.$$

A GCM

$$(5) \quad y = f(x) = f_0(x_0, \dots, x_s) + \sum_{n=1}^s f_n(x_0, \dots, x_s) \varepsilon_n$$

satisfies the conditions  $f_k \in C^1$ ,  $k = 0, \dots, s$  and

$$(6) \quad \chi[dy]/\chi[dx] = [dy_0, \dots, dy_s]/[dx_0, \dots, dx_s] = \text{inv}(x),$$

independent of the direction  $dx$ .

I. The book begins with the CM corresponding to the Euclidean plane as well as to the planes with Clifford's, Boole's and Study's metric, respectively. Then the author introduces the space  $\Pi_\sigma$  and a group of invariant transformations with respect to (3) (which generalizes the group of rotations, Lorentz's group, etc.) and defines the GCM.

II. The author studies the algebra of complexes with the central problem, the LC. The places are interpreted as matrices  $\{\varepsilon_0\} = \{1\}$ ,  $\{\varepsilon_n\} = \{\gamma_{nk}^j\}$  ( $\gamma_{n0}^j = \delta_n^j$ ), so that  $\{x\} = \{x_0 \delta_k^j + \gamma_{nk}^j x_k\}$ . Many LC are discussed. The division in the algebra leads to the characteristic of  $x$  denoted again by  $\chi[x]$  since it will coincide with the metric form, and which satisfies

$$(7) \quad \chi[x] = \det \{x\}.$$



By means of the „numerical values” of the places (i.e. of the systems of  $s + 1$  numbers  $\alpha_{0j} = 1, \alpha_{1j}, \dots, \alpha_{sj}$  such that if one replaces  $\epsilon_n$  by  $\alpha_{nj}$  the LC is verified) the „ $\sigma$ -functions” are defined

$$(8) \quad \sigma_j [x] = x_j + \sum_{n=0}^s \alpha_{nj} x_n$$

and the following two cases considered: 1°. Suppose that there are  $s + 1$   $\sigma$ -functions  $\tau_j [x]$ ,  $j = 0, \dots, s$ , and that  $\Delta = |\alpha_{kj}| \neq 0$ . Then the characteristic (7)  $\chi[x] = \prod_{j=0}^s \sigma_j [x]$  and the LC is called „solvable”. There exist the „quasi-units”  $E_j$  so that

$$(9) \quad x = \sum_{j=0}^s E_j \sigma_j [x].$$

2°. If the characteristic (7) has the form (3) (i.e. there exist multiple  $\sigma$ -functions) the LC is called „solvable at the limit” and one obtains

$$(9') \quad x = \sum_{h=0}^p E_h \sigma_h [x] + R [x],$$

with a complementary complex  $R[x]$ . Starting with a metric form (3) the author constructs the LC so that the metric form (3) coincides with the characteristic of  $x$  (7). In the case 2° he first solves the problem for orthogonal LC and then he uses an affine transformation.

In III he deals with the AF with the aim of obtaining by this way an explicit construction for the GCM. A mapping  $f$  is called monogenic if replacing  $x_n = x - \sum_{n=0}^s x_n$  in  $f$  one obtains a function of  $x$  which does not depend on  $x_n$ . One supposes  $f \in C^1$  and deduces the generalization of the Cauchy-Riemann system. A mapping  $f \in C^1$  is called AF if  $f(x)dx = d\Phi(x)$ ; it is monogenic and there exists the derivative  $f' = \lim_{\Delta x \rightarrow 0} [(f(x + \Delta x) - f(x))/\Delta x]$ ,  $\chi[\Delta x] \neq 0$ . An AF  $f$  is a GCM and (6)  $\text{inv} = \chi[f]$ . Further the author constructs the representations (9) and (9'), the main tool being the  $\sigma$ -functions which are determined as the eigen-values of the matrix  $\{x\}$ . The direct value of an AF is obtained by replacing  $z$  in an AF of the complex variable  $z$  by a complex  $x$  (1) and calculating with the LC. The author establishes the explicit form of the direct value of  $f$ :

$$(10) \quad f(x) = \sum_{j=0}^s E_j f(\sigma_j [x])$$

in the case 1°, and

$$(10') \quad f(x) = \sum_{h=0}^p E_h d_n f(\sigma_h [x]),$$

$$d_h = 1 + R \frac{d}{d\sigma_h} + \frac{1}{2} R^2 \frac{d^2}{d\sigma_h^2} + \dots + \frac{1}{(N_h - 1)} R^{N_h - 1} \frac{d^{N_h - 1}}{d\sigma_h^{N_h - 1}},$$

where  $N_h \leq n_h$  is the multiplicity of  $\tau_h$  as a root of the minimal polynomial satisfied by  $x$ , in the case 2°. Then he demonstrates the contour integral representation of the direct value of an AF and of its derivatives of an arbitrary order. The case of AF of several complexes is also discussed. Further the direct value theory is applied to the reduction of some partial differential operators to ordinary differential operators. More generally, the author studies the „basic representation of AF”, by considering for each  $\sigma$ -function a corresponding AF  $\varphi$  and putting

$$1^\circ F < \varphi_0, \dots, \varphi_s > (x) = \sum_{j=0}^s E_j \varphi_j (\sigma_j [x]) \quad \text{or} \quad 2^\circ F < \varphi_0, \dots, \varphi_s > (x) = \sum_{h=0}^p E_h \varphi_h (\sigma_h [x]),$$

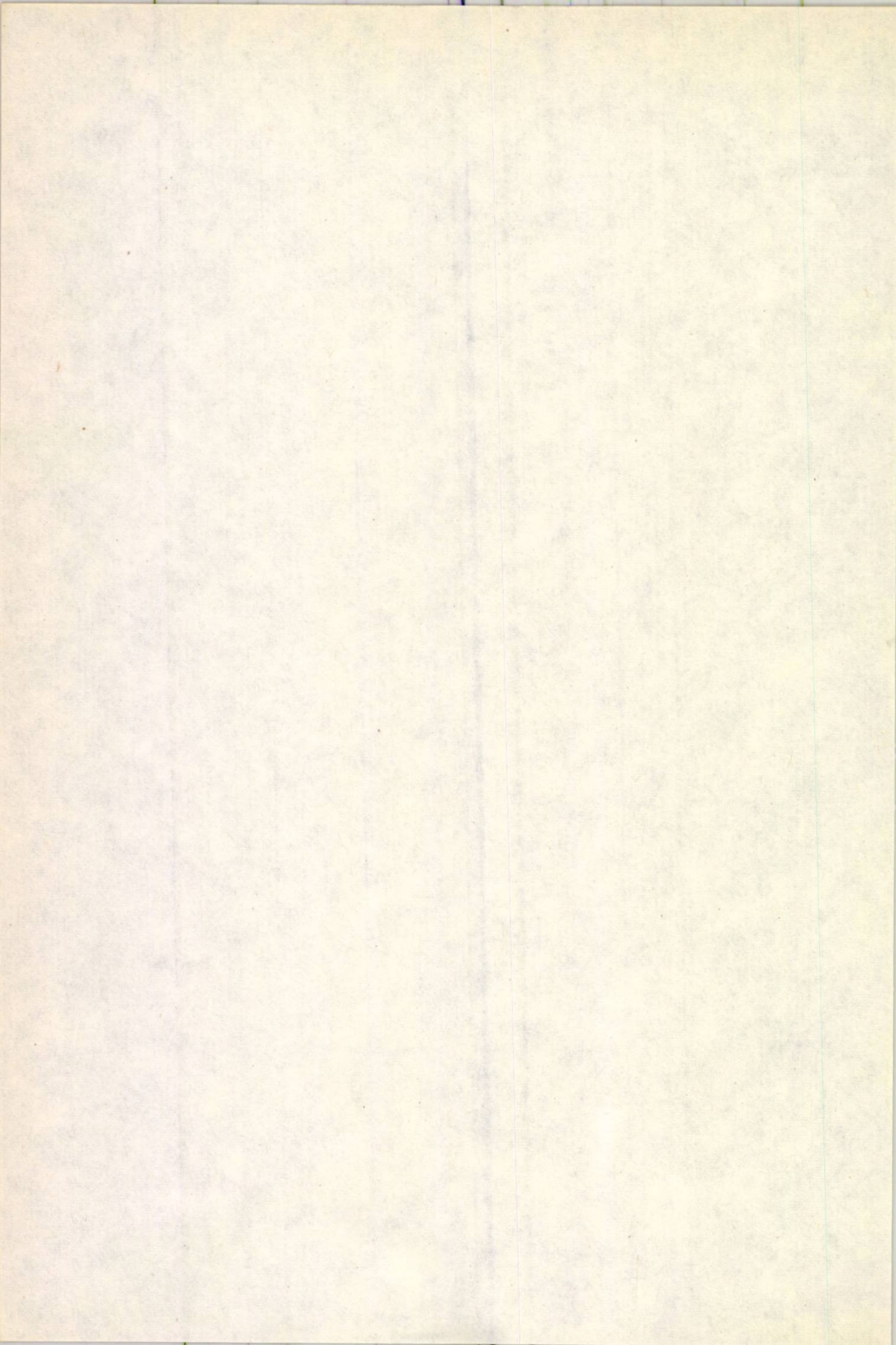
with the condition  $Fdx = d\Phi$ . He deduces explicit basic representation formulae for AF and finally obtains the explicit form for the GCM.

The monograph contains an extremely rich and valuable collection of examples which clarify the theory and in many cases are important for different applications.

Cabiria Andreian Cazacu









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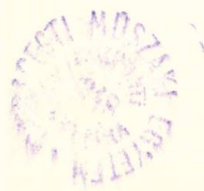
**SECȚIA I**

**MATEMATICĂ  
MECANICĂ TEORETICĂ  
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**MATEMATICĂ  
SECȚIA I MECANICĂ TEORETICĂ  
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**1979**

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# ВКЛАД В ЗАДАЧИ ОБ АСИМПТОТИКЕ ЧИСЛА СОБСТВЕННЫХ ЗНАЧЕНИЙ

## I. ОБ ОЦЕНКЕ НЕКОТОРЫХ РАЗЛОЖЕНИЙ ПО ФУНКЦИЯМ БЕССЕЛЯ

Д. МАНЖЕРОН, А. Ф. ШЕСТОПАЛ, У. Д'АМБРОЗИО и С. НЯНГ

Посвящается светлой памяти  
академика И. Н. ВЕКУА,  
Г. ВРЭНЧАНУ и М. ПИКОНЕ

0. В ряде вопросов математического анализа, теории чисел и математической физики представляет большой интерес получить асимптотику при больших  $\lambda$  рядов следующего вида

$$(1) \quad D_n^{\vec{v}}(\lambda, \vec{\rho}) = \frac{\lambda^{\frac{n}{4}}}{\sqrt{\det A}} \sum_{i \in Z^n} \frac{I_n(2\pi \sqrt{\lambda(A^{-1} \vec{s}_i \vec{s}_i)})}{(A^{-1} \vec{s}_i, \vec{s}_i)^{\frac{n}{4}}} e^{2\pi i(\vec{\rho}, \vec{s}_i)},$$

где  $Z$  — группа целых чисел,  $I_n(x)$  — функция Бесселя первого рода  $n$ -ого порядка,  $A^{-1}$  — симметричная матрица порядка  $(n \times n)$ ,  $\vec{s}_i = \vec{s} + \vec{v}$ ,  $(A^{-1} \vec{y}, \vec{y})$  — положительно определенная квадратичная форма на  $n$ -мерном евклидовом пространстве  $E^n$  и, наконец,  $\vec{v} \in E^n$ ,  $\vec{\rho} \in E^n$  — фиксированные вектора.

Поскольку  $D_n^{\vec{v}}(\lambda, \vec{\rho})$  периодическая по каждой компоненте  $\rho_i$  вектора  $\vec{\rho}$  функция, то будем предполагать, что  $\rho_i \in [0, 1)$ , причем  $(\sqrt{\lambda \det A^{-1}} \pm \rho_i) \notin Z$ , если  $v_i \in Z$ .

Оказывается, что ряды типа [1] суммируемы как по „строкам“, так и по „столбцам“ и их оценка связана при  $\lambda \rightarrow \infty$ ,  $\vec{v} \in Z^n$  с оценкой числа целых точек в эллипсоиде, заданном уравнением

$$(2) \quad (A \vec{y}, \vec{y}) = \lambda, \quad \vec{y} = \vec{x} + \vec{\rho}, \quad \vec{x} \in E^n.$$

В случае  $\vec{v} \notin Z^n$  асимптотика рядов (1) при больших  $\lambda$  существенно отличается от асимптотики при  $\vec{v} \in Z^n$  и связана с оценкой при больших  $\lambda$  тригонометрических сумм вида

$$(3) \quad \sum_{a \leq p < b} e^{i \sqrt{\lambda} U(p)},$$

где, как обычно,  $[x]$  обозначает целую часть  $x$ , а суммирование ведется по всем целым  $p$  из интервала  $[a, b]$ .



1. Прежде чем переходить к оценке ряда (1) договоримся о некоторых обозначениях и упрощениях. Во-первых, при  $\vec{v} = \vec{0}$  индекс  $\vec{0}$  формуле (1) будем опускать, во-вторых, иногда скалярное произведение  $(A^{-1}\vec{y}, \vec{y})$  будем снабжать внизу индексом  $n$ , если  $\vec{y} \in E^n$ , и, в-третьих, вектор  $\vec{y}' \in E^{n+1}$  будем записывать в виде  $\vec{y}' = (\vec{y}, y_{n+1})$ , где  $\vec{y} \in E^n$ ,  $y_{n+1} \in E$ .

**Теорема 1.** Число целых точек в эллипсоиде, заданном уравнением (2) равно  $D_n(\lambda, \bar{\rho})$ .

**Доказательство.** Доказательство будем вести индукцией по  $n$ . Итак, пусть  $n = 1$ . Тогда, учитывая, что

$$(4) \quad I_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin z,$$

получаем

$$(5) \quad D_1(\lambda, \rho) = \frac{1}{\pi} \sum_{s \in \mathbb{Z}} \frac{\sin 2\pi s \sqrt{\lambda A^{-1}}}{s} \cos 2\pi \rho s.$$

Используя хорошо известное разложение в ряд Фурье

$$(6) \quad [x] + \frac{1}{2} = \frac{1}{2\pi} \sum_{s \in \mathbb{Z}} \frac{\sin 2\pi s x}{s}$$

находим

$$(7) \quad D_1(\lambda, \rho) = [-\rho + \sqrt{\lambda A^{-1}}] - [-\rho - \sqrt{\lambda A^{-1}}].$$

Стало быть, при  $n = 1$  теорема верна.

Обозначая, далее, через  $B$  симметричную положительно-определенную квадратную матрицу размера  $(n+1) \times (n+1)$  с элементами  $a_{ij}$  ( $1 \leq i, j \leq n+1$ ). Пусть  $A$  матрица размера  $n \times n$  с элементами  $a_{ij}$  ( $1 \leq i, j \leq n$ ), а  $\vec{a}_{n+1}$  вектор из  $E^n$  с компонентами  $a_{n+1,i}$  ( $1 \leq i \leq n$ ). В этих обозначениях уравнение  $(n+1)$ -мерного эллипсоида

$$(8) \quad (B\vec{y}', \vec{y}')_{n+1} = \lambda$$

можно записать в виде

$$(9) \quad (A\vec{y}, \vec{y})_n + 2z(\vec{a}_{n+1}, \vec{y})_n + z^2 a_{n+1, n+1} = \lambda,$$

где  $\vec{y} = \vec{x} + \vec{\rho}$ ,  $z = \rho_{n+1} + x_{n+1}$ .

Уравнение (9) можно еще переписать так

$$(10) \quad (A\vec{v}, \vec{v}) = \lambda',$$

где

$$(11) \quad \begin{aligned} \vec{v} &= \vec{x} + \vec{\rho} + z\vec{\mu}, & \lambda' &= \lambda - z^2 p, \\ p &= a_{n+1, n+1} - (A\vec{\mu}, \vec{\mu}), & \mu &= A^{-1}\vec{a}_{n+1}. \end{aligned}$$

Матрица  $A$  положительно-определена и симметрична. Количество целых точек в эллипсоиде, заданном уравнением (10),



$$(12) \quad D_{n+1}(\lambda, \bar{\rho}') = \sum_{-\rho_{n+1} - \sqrt{\frac{\lambda}{p}} < z_{n+1} < -\rho_{n+1} + \sqrt{\frac{\lambda}{p}}} D_n(\lambda', \bar{\rho} + z\bar{\mu}).$$

По формуле суммирования Пуассона [2] получим

$$(13) \quad D_{n+1}(\lambda, \bar{\rho}') = \sum_{\substack{\bar{z} \in \mathbb{Z} \\ -\sqrt{\frac{\lambda}{p}}}}^{\sqrt{\frac{\lambda}{p}}} D_n(\lambda', \bar{\rho} + z\bar{\mu}) e^{2\pi i s_{n+1}}.$$

Подставляя в формулу (13) выражение для  $D_n(\lambda', \bar{\rho} + z\bar{\mu})$ , которое в силу индуктивного предположения определяется формулой (1), находим

$$(14) \quad D_{n+1}(\lambda, \bar{\rho}') = \sum_{\bar{z} \in \mathbb{Z}^{n+1}} \frac{e^{2\pi i(\bar{s}', \bar{\rho}')}}{\sqrt{\det A}} \int_{-\sqrt{\frac{\lambda}{p}}}^{\sqrt{\frac{\lambda}{p}}} A_{\bar{s}}^{-1}(z) e^{2\pi i z \{\bar{s}, \bar{\mu} - s_{n+1}\}} dz,$$

где  $\bar{s}' = (\bar{s}, s_{n+1})$ , а

$$(15) \quad A_{\bar{s}}^{-1}(z) = \frac{(\lambda')^{\frac{n}{4}}}{(A^{-1}\bar{s}, \bar{s})^{\frac{n}{4}}} I_{\frac{n}{2}}(2\pi \sqrt{\lambda'}(A^{-1}\bar{s}, s)).$$

Поскольку [10]

$$(16) \quad 4\sqrt{\lambda} \int_{-1}^1 (1-x^2)^{\frac{\nu}{2}} \frac{I_{\nu}(2\pi \lambda \sqrt{\lambda} \sqrt{1-x^2})}{b^{\nu}} e^{2\pi i a \sqrt{\lambda} x} dx = \frac{I_{\nu+\frac{1}{2}}(2\pi \sqrt{\lambda} \sqrt{a^2+b^2})}{(\sqrt{a^2+b^2})^{\nu+\frac{1}{2}}},$$

то

$$(17) \quad D_{n+1}(\lambda, \bar{\rho}') = \frac{\lambda^{\frac{n+1}{4}}}{\sqrt{p \det A}} \sum_{\bar{z} \in \mathbb{Z}^{n+1}} \frac{I_{\frac{n+1}{2}}(2\pi \sqrt{\lambda} Q_{\bar{z}})}{Q_{\bar{z}}^{\frac{n+1}{4}}} e^{2\pi i(\bar{\rho}', \bar{s}')} ,$$

где

$$(18) \quad Q_{\bar{z}} = (A^{-1}\bar{s}, \bar{s}) + \frac{[s_{n+1} - (\bar{s}, \bar{\mu})]^2}{p}.$$

Теперь остается  $Q_{\bar{z}}$  выразить через  $B^{-1}\bar{s}'$ . С этой целью рассмотрим систему уравнений

$$(19) \quad B\bar{y}' = \bar{s}'.$$

Из этой системы находим

$$(20) \quad A\bar{y} = \bar{s} - y_{n+1} \bar{a}_{n+1},$$

так что

$$(21) \quad \bar{y} = A^{-1}\bar{s} - y_{n+1} A^{-1} \bar{a}_{n+1}.$$

Подставляя  $\vec{y}$  в последнее уравнение системы (19), получаем

$$(22) \quad y_{n+1} = \frac{s_{n+1} - (A^{-1}\vec{s}, \vec{a}_{n+1})}{a_{n+1, n+1} - (A^{-1}\vec{a}_{n+1}, \vec{a}_{n+1})}.$$

Таким образом

$$(23) \quad (B^{-1}\vec{s}', \vec{s}') = (\vec{y}, \vec{s}) + y_{n+1} s_{n+1} = (A^{-1}\vec{s}, \vec{s}) + \frac{[s_{n+1} - (\vec{\mu}, \vec{s})]^2}{p} = Q_{\vec{s}}.$$

Далее

$$(24) \quad \begin{aligned} p \det A &= a_{n+1, n+1} \det A - \det A (\vec{a}_{n+1}, A^{-1}\vec{a}_{n+1}) = \\ &= a_{n+1, n+1} \det A + (-1)^{n+n+1} \sum_{i,j=1}^n A_{ij} a_{n+1, i} a_{n+1, j}, \end{aligned}$$

где  $A_{ij}$  алгебраическое дополнение элемента  $a_{ij}$  матрицы  $A$ . Но тогда по теореме Лапласа  $p \det A = \det B$ . Теорема доказана.

Из разложения функции Бесселя в ряд

$$(25) \quad I_m(z) = \left(\frac{z}{2}\right)^m \sum_{k=0}^{\infty} (-1)^k \left(\frac{z}{2}\right)^{2k} \frac{1}{k! \Gamma(m+k+1)}$$

усматриваем, что член соответствующий  $\vec{s} = \vec{0}$  в разложении (1) равен

$$(26) \quad \frac{\lambda^{\frac{n}{2}}}{2^{\frac{n}{2}} \Gamma\left(\frac{n+2}{2}\right) \det A},$$

что совпадает с первым членом асимптотики числа целых точек в эллипсоиде, заданном уравнением (2), поскольку это выражение есть объем эллипсоида. Что касается оставшихся членов, то в силу хорошо известных асимптотических разложений [8]

$$(27) \quad I_\nu(z) = \sqrt{\frac{2}{\pi z}} \left\{ \cos\left(z - \frac{\pi}{2}\nu - \frac{\pi}{4}\right) + O\left(\frac{1}{z}\right) \right\},$$

каждый из них имеет порядок  $O(\lambda^{\frac{n-1}{4}})$ .

Разумеется отсюда нельзя сделать вывод, что и весь ряд имеет такой же порядок, поскольку он не сходится абсолютно. Однако оценку типа  $\Omega$  отсюда можно получить.

Что же касается  $O$ -проблемы, то при  $n \geq 5 =$  порядок остатка, как показывают результаты работы [4] будет  $O(\lambda^{\frac{n}{2}-1})$ , причем эта оценка точна по порядку  $\lambda^{\frac{n}{2}}$ . Если же  $n \leq 4$ , то здесь нет точных по порядку оценок, несмотря на очень большое число результатов в этом направлении (см. обзор в [5]). Во всяком случае остаток имеет порядок  $O(\lambda^{\frac{n-1}{2}})$ . Итак



$$(28) \quad D_n(\lambda, \rho) = \frac{\lambda^{\frac{n}{2}}}{2^{\frac{n}{2}} \Gamma\left(\frac{n+2}{2}\right) \det A} + o\left(\lambda^{\frac{n-1}{2}}\right).$$

2. Теперь рассмотрим случай  $\vec{v} \notin \vec{0}$ . Заметим, что случай  $\vec{v} \in Z^n$  не дает ничего нового, поскольку в этом случае

$$(29) \quad D_n^{\vec{v}}(\lambda, \vec{\rho}) = e^{-2\pi i(\vec{\rho}, \vec{v})} D_n(\lambda, \vec{\rho}).$$

Таким образом, следует рассмотреть лишь случай  $\vec{v} \notin Z^n$ .

Т е о р е м а 2. При достаточно большом  $\lambda$

$$(30) \quad |D^{\vec{v}}(\lambda, \vec{\rho})| \leq D_n(\lambda, \rho),$$

причем знак равенства может достигаться лишь при  $\vec{v} \in Z^n$ .

Д о к а з а т е л ь с т в о. Установим сначала аналог формулы (12), а именно покажем, что

$$(31) \quad D_{n+1}^{\vec{v}}(\lambda, \vec{\rho}') = \sum_{-\rho_{n+1} - \sqrt{\frac{\lambda}{p}} < z_{n+1} < -\rho_{n+1} + \sqrt{\frac{\lambda}{p}}} D_n^{\vec{v}}(\lambda', \vec{\rho} + z\vec{\mu}), e^{2\pi i\{(\vec{v}, \vec{\mu}) - v_{n+1}\}},$$

где все обозначения такие же, как и в (11).

Действительно по формуле суммирования Пуассона получаем

$$(32) \quad D_{n+1}^{\vec{v}}(\lambda, \vec{\rho}') = \sum_{s_{n+1} \in Z} \int_{-\sqrt{\frac{\lambda}{p}}}^{\sqrt{\frac{\lambda}{p}}} D_n^{\vec{v}}(\lambda', \vec{\rho} + z\vec{\mu}) \exp 2\pi i \{z[(\vec{v}, \vec{\mu}) - v_{n+1}] - s_{n+1}(z - \rho_{n+1})\} dz.$$

Подставляя в формулу (32) выражение (1), находим что

$$(33) \quad D_{n+1}^{\vec{v}}(\lambda, \vec{\rho}') = \sum_{\vec{s} \in Z^{n+1}} \frac{\exp 2\pi i(\vec{\rho}', \vec{s}')}{\sqrt{\det A}} \int_{-\sqrt{\frac{\lambda}{p}}}^{\sqrt{\frac{\lambda}{p}}} A_{\vec{s}}(z) \exp 2\pi i \{(\vec{s}, \vec{v}) - z(s_{n+1} + v_{n+1})\} dz,$$

что совпадает с формулой (14) с точностью до обозначений.

Теперь доказательство можно провести методом индукции. При  $n = 1$  получаем

$$(34) \quad D_1^{\vec{v}}(\lambda, \rho) = \frac{1}{\pi} \sum_{s \in Z} \frac{\sin 2\pi(s + v)\sqrt{\lambda A^{-1}}}{s + v} e^{2\pi i \rho s}.$$

Выражение (34) можно просуммировать с помощью разложения в ряд Фурье

$$(35) \quad \sum_{s \in \mathbb{Z}} \frac{\exp \{2\pi i (s + v) x\}}{s + v} = \pi \frac{\exp \{2\pi i (2[x] + 1)\}}{\sin v\pi},$$

так что  $D^v(\lambda, \rho) = O(1)$ , если  $v \notin \mathbb{Z}$ , в то время как  $D_1(\lambda, \rho) = O(|\sqrt{\lambda}|)$ . Тогда по формуле (31) получим (30).

Более того, если хотя бы одно из  $v_i$  не целое, то

$$(36) \quad D_n^{\vec{v}}(\lambda, \bar{\rho}) = O(\lambda^{\frac{n-1}{2}}),$$

в то время как при  $\vec{v} \in \mathbb{Z}^n$

$$(37) \quad D_n^{\vec{v}}(\lambda, \bar{\rho}) = O(\lambda^{\frac{n}{2}}).$$

**С л е д с т в и е .** Пусть  $\vec{v}_k$  вектор, полученный из  $\vec{v}$  путем замены  $k$  нецелых координат нулями. Тогда  $D_n^{\vec{v}_k}(\lambda, \bar{\rho})$  и  $\lambda^{-\frac{k}{2}} D_n^{\vec{v}_k}(\lambda, \bar{\rho})$  имеют одинаков при  $\lambda \rightarrow \infty$  порядок роста.

**Т е о р е м а 3.** Пусть

$$(38) \quad I_{n+1} = \int_D D_{n+1}^{\vec{v}_n}(\lambda, \bar{\rho}') d\vec{v}',$$

где  $\vec{v}_n = (0, 0, \dots, 0, v_{n+1})$ ,  $\vec{v}' = (\vec{v}, v_{n+1})$  а  $D$  область любого ограниченного многогранника. Тогда

$$(39) \quad I_{n+1} = O(\lambda^{\frac{n}{2}}),$$

если  $s_{n+1} + v_{n+1}$  не обращается в 0 в области  $D$  при любом  $s_{n+1} \in \mathbb{Z}$ .

**Д о к а з а т е л ь с т в о .** По формуле (31) имеем

$$(40) \quad I_{n+1} = \sum_{-c_{n+1} - \sqrt{\lambda} \leq v_{n+1} \leq -c_{n+1} + \sqrt{\lambda}} \int D_n(\lambda', \bar{\rho} + z\bar{\mu}) e^{-2\pi i z v_{n+1}}.$$

Учитывая формулу (28), получаем

$$(41) \quad 2^{\frac{n}{2}} \Gamma\left(\frac{n+2}{2}\right) \lambda^{-\frac{n}{2}} I_{n+1} = \sum \int (\lambda' \lambda^{-1})^{\frac{n}{2}} \frac{e^{-2\pi i z v^1} - e^{-2\pi i z v^2}}{z} d\vec{v} + O(1),$$

где  $v^i$  линейные формы по  $v$ , которые в рассматриваемой области имеют одинаковый знак, ни одна из них не обращается в 0 и  $|v^i| < 1$ . Пусть сначала  $n$ -четное. Тогда

$$(42) \quad (\lambda' \lambda^{-1})^{\frac{n}{2}} = 1 + \sum \alpha_i \frac{z^{2i} p^i}{\lambda^i},$$

причем сумма в формуле (42) конечная. Учитывая, что при  $|\alpha| < 1$  и целом  $j \geq 1$

$$(43) \quad \sum_{k=1}^{[\sqrt{\lambda}]} k^{2j-1} \sin 2\pi k \alpha = O(\lambda^{j-\frac{1}{2}})$$



получаем, что

$$2^{\frac{n+1}{2}} \det A \Gamma\left(\frac{n+2}{2}\right) I_{n+1}$$

при  $\lambda \rightarrow \infty$  имеет своим пределом выражение

$$(44) \quad \frac{1}{2\pi} \sum_{z \in Z} \frac{\sin 2\pi z v^2 - \sin 2\pi z v^1}{z} = [v^2] - [v^1],$$

которое в силу предположений о формах  $v_i$  равно 0. Итак

$$(45) \quad I_{n+1} = o(\lambda^{\frac{n}{2}})$$

при четном  $n$ . Если же  $n$  нечетное, то по формуле (31)

$$(46) \quad I_{n+1} = O(\sum I_n) = O(\sqrt{\lambda} I_n) = o(\lambda^{\frac{n}{2}}).$$

3. Полученные оценки можно применить для получения точного второго члена асимптотики функции  $N(\lambda)$  [1] в задаче Неймана и Дирихле для оператора Лапласа в области зеркально-симметричного многогранника, т. е. многогранника, отражения которого относительно гиперплоскостей, на которых лежат его грани, ведут к заполнению (без перекрытий) всего пространства  $E^n$ . Все эти фигуры можно найти с помощью конечных групп Костера [3].

Оказывается, что функция  $N(\lambda)$ , связанная с граничной задачей

$$(47) \quad \begin{aligned} -\Delta u &= \lambda u, & x &\in D \\ \frac{\partial u}{\partial \vec{n}} &= 0, & x &\in \partial D, \end{aligned}$$

где  $D$  область вышеупомянутого многогранника, а  $\vec{n}$  нормаль к  $\partial D$  (точнее к граням), может быть представлена в виде

$$(48) \quad N(\lambda) = \frac{\lambda^{\frac{n}{4}}}{(2\pi)^{\frac{n}{2}}} \sum_{i \in Z} \sum_{\alpha \in W} \int_D \frac{I_n(\sqrt{\lambda} \|A\vec{s} + \alpha(\vec{x}) - \vec{x}\|)}{\|A\vec{s} + \alpha(\vec{x}) - \vec{x}\|^{\frac{n}{2}}} d\vec{x}.$$

В формуле (48)  $A$  — характеристическая матрица многогранника  $D$ , а  $W$  — группа отражений Вейля.

Подробно об этом, а также о применимости наших рядов в теории чисел, будет сказано в последующих публикациях авторов.

Одновременно готовится, в связи с рядом результатов авторов в областях математического анализа [11] — [14] и математической физики [15], другая серия совместных или же отдельных публикаций авторов и некоторых из их сотрудников, углубляющих некоторые разветвления богатейшего научного наследия оставленного незабвенными академиками И. Н. Векуа, Г. Врэнчану и М. Пиконе [16] — [18].

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## CONTRIBUȚII LA STUDIUL PROBLEMEI DE EVALUARE ASIMPTOTICĂ A NUMĂRULUI DE VALORI PROPRII

### I. Evaluarea unor dezvoltări după funcții Bessel

(Rezumat)

În această primă parte a unui studiu consacrat evaluării asimptotice a numărului de valori proprii, se stabilește o evaluare asimptotică a seriilor de forma (1), pentru valori mari ale lui  $\lambda$ . Astfel de serii intervin în probleme de analiză matematică, teoria numerelor și fizică matematică.



## SOME REMARKS ON THE PATH FIBRATION OF A TOPOLOGICAL SPACE

BY

I. POP

1. In this Note the construction of the covering spaces, given by the classification theorem [1, p. 82], is generalized, thus obtaining some Hurewicz fiber spaces.

Let  $X$  be a connected locally path-connected space. It is known that to every connected locally path-connected covering space

$$(1) \quad p: \tilde{X} \longrightarrow X$$

it corresponds a subgroup  $H = p_{\#}\pi_1(\tilde{X}, \tilde{x}_0)$  of the group  $\pi_1(X, p(\tilde{x}_0))$  and an open covering  $\mathcal{U} = \{U\}$  of the space  $X$ , such that

$$(2) \quad \pi_1(\mathcal{U}, x_0) \subset H.$$

Conversely, to any pair  $(\mathcal{U}, H)$  satisfying the condition (2) it can be associated a covering projection (1), such that  $p_{\#}\pi_1(\tilde{X}, \tilde{x}_0) = H$ .

The space  $\tilde{X}$  is constructed as a factor space of the space  $PX$  of the paths in  $X$ , with the origin in a point  $x_0 \in X$ . In the proof that the obtained projection is a covering map there is used, in an essential way, the existence of an open covering  $\mathcal{U}$  of the space  $X$ , satisfying the condition (2).

It is interesting to study the map  $p: \tilde{X} \rightarrow X$  without the condition (2). We consider a more general situation than that exposed above.

2. Let  $X$  be a path-connected space and let  $x_0 \in X$ . Consider the standard contractible path fibration  $P: PX \rightarrow X$ , where  $PX$  is considered with compact open topology, and  $P(\omega) = \omega(1)$ .

Let  $\rho$  be an equivalence relation in the space  $PX$ , such that there are satisfied the following conditions:

a) If  $\omega, \omega' \in PX$ , then  $\omega\rho\omega' \Rightarrow \omega(1) = \omega'(1)$ ;

b) If  $\omega, \omega' \in PX$ , such that  $\omega\rho\omega'$  and  $\theta: I \rightarrow X$  is a path, such that  $\theta(0) = \omega(1) = \omega'(1)$ , then  $(\omega * \theta) \underset{2}{\underset{t+1}{\rho}} (\omega' * \theta) \underset{2}{\underset{t+1}{\rho}}$ , for every  $t \in I$ , where,

for a path  $\mu: I \rightarrow X$ , we denote by  $\mu_t$  the path  $\mu_t(t') = \mu(tt')$ .

Denote by  $\langle \omega \rangle$  the equivalence class of the path  $\omega \in PX$  and let  $E$  be the set of equivalence classes.



Let  $\pi: PX \rightarrow E$  be the canonical projection:  $\pi(\omega) = \langle \omega \rangle$  and consider  $E$  with the quotient topology.

By the condition a), we can consider the map  $p: E \rightarrow X$  defined by  $p(\langle \omega \rangle) = \omega(1)$ .

**Theorem.** *The map  $p: E \rightarrow X$  is a Hurewicz fibration.*

**Proof.** First, we prove that  $p$  is a continuous map. For this, we consider the following commutative diagram

$$\begin{array}{ccc} PX & \xrightarrow{P} & X \\ \pi \uparrow & \nearrow p & \\ E & & \end{array}$$

If  $D$  is an open set in  $X$ , then  $\pi^{-1}(p^{-1}(D)) = P^{-1}(D)$  and, since  $P$  is a continuous map, it follows that  $P^{-1}(D)$  is an open set in  $PX$ . Thus,  $p^{-1}(D)$  is an open set in  $E$ .

Now, we construct a lifting function for the map  $p$ . Since  $P: PX \rightarrow X$  is a Hurewicz fibration, there exists a lifting function for it. Namely, if

$$\text{Hur } PX = \{(\omega, \theta) \in PX \times X^I / \omega(1) = \theta(0)\},$$

then a lifting function for  $P$  is  $\lambda: \text{Hur } PX \rightarrow (PX)^I$ , defined by  $\lambda(\omega, \theta)(t) = (\omega * \theta)_{\frac{t+1}{2}}$ . Now, let

$$\text{Hur } E = \{(\langle \omega \rangle, \theta) \in E \times X^I / \omega(1) = \theta(0)\}.$$

By the condition b), the following map  $\tilde{\lambda}: \text{Hur } E \rightarrow E^I$ , with  $\tilde{\lambda}((\langle \omega \rangle, \theta))(t) = \langle (\omega * \theta)_{\frac{t+1}{2}} \rangle$ , is well defined.

Now, we shall prove that  $\tilde{\lambda}$  is a continuous map. Remark that the following diagram is commutative:

$$\begin{array}{ccc} \text{Hur } PX & \xrightarrow{\lambda} & (PX)^I \\ \pi \times 1 \downarrow & & \downarrow \pi^I \\ \text{Hur } E & \xrightarrow{\tilde{\lambda}} & E^I \end{array}$$

where  $(\pi^I)(\varphi) = \pi \circ \varphi$  and  $(\pi \times 1)((\omega, \theta)) = (\pi(\omega), \theta) = (\langle \omega \rangle, \theta)$ . In fact  $\pi^I \lambda((\omega, \theta))(t) = \pi(\lambda((\omega, \theta)))(t) = \pi((\omega * \theta)_{\frac{t+1}{2}}) = \langle (\omega * \theta)_{\frac{t+1}{2}} \rangle = \tilde{\lambda}((\langle \omega \rangle, \theta))(t) = \tilde{\lambda}(\pi \times 1)((\omega, \theta))(t)$ , that is  $\pi^I \lambda((\omega, \theta)) = \tilde{\lambda}(\pi \times 1)((\omega, \theta))$ .

Now, let  $V$  be an open set in  $E^I$ . Then,  $(\pi \times 1)^{-1}(\tilde{\lambda}^{-1}(V)) = \lambda^{-1}((\pi^I)^{-1}(V))$  is an open set and it follows that  $\tilde{\lambda}^{-1}(V)$  is an open set, that is  $\tilde{\lambda}$  is a continuous map.

We prove that  $\tilde{\lambda}$  is a lifting function for  $p$ . In fact:

1) For  $(\langle \omega \rangle, \theta) \in \text{Hur } E$ , we have  $\tilde{\lambda}((\langle \omega \rangle, \theta))(0) = \langle (\omega * \theta)_1 \rangle$ .



But,  $(\omega * \theta)_{1/2}(t) = (\omega * \theta)(t/2) = \omega(t)$ , that is  $\langle (\omega * \theta)_{1/2} \rangle = \langle \omega \rangle$ . Thus, we obtain  $\tilde{\lambda}((\langle \omega \rangle, \theta))(0) = \langle \omega \rangle$ .

2) For  $(\langle \omega \rangle, \theta) \in \text{Hur } E$ , obtain  $p\tilde{\lambda}((\langle \omega \rangle, \theta))(t) = p(\langle (\omega * \theta)_{\frac{t+1}{2}} \rangle) = (\omega * \theta)_{\frac{t+1}{2}}(1) = (\omega * \theta)\left(\frac{t+1}{2}\right) = \theta(t)$ .

This finishes the proof of the theorem.

**Corollary (Example 1).** Let  $X$  be a path-connected space and let  $H$  be a subgroup of the group  $\pi_1(X, x_0)$ ,  $x_0 \in X$ .

In the path space  $PX$  consider the equivalence relation  $\omega \sim \omega' \Leftrightarrow \omega(1) = \omega'(1)$  and  $[\omega * \omega'^{-1}] \in H$ . Denoting by  $E$  the space of equivalence classes and by  $p: E \rightarrow X$  the map  $p(\langle \omega \rangle) = \omega(1)$ , one obtains a Hurewicz fibration.

The proof of this Corollary follows from the Theorem. The condition a) is obviously satisfied and the condition b) is verified as follows: if  $\omega, \omega' \in PX$  such that  $\omega \sim \omega'$ , and  $\theta: I \rightarrow X$  is a path with  $\theta(0) = \omega(1) = \omega'(1)$ , then  $(\omega * \theta)_{\frac{t+1}{2}}(1) = (\omega' * \theta)_{\frac{t+1}{2}}(1) = \theta(t)$  and  $(\omega * \theta)_{\frac{t+1}{2}} * (\omega' * \theta)_{\frac{t+1}{2}}^{-1} \simeq \omega * \omega'^{-1} \text{ rel } \dot{I}$ , by the homotopy

$$F(t', t'') = ((\omega * \theta)_{\frac{t'+1}{2}} * (\omega' * \theta)_{\frac{t'+1}{2}}^{-1})(t').$$

Therefore,  $[(\omega * \theta)_{\frac{t+1}{2}} * (\omega' * \theta)_{\frac{t+1}{2}}^{-1}] = [\omega * \omega'^{-1}] \in H$ , that is  $(\omega * \theta)_{\frac{t+1}{2}} \sim (\omega' * \theta)_{\frac{t+1}{2}}$ .

In order to illustrate the above corollary, consider  $X$  as a countable product of 1-spheres and let  $H = \{0\} \subset \pi_1(X, x_0)$ . Then, the projection  $p: E \rightarrow X$ , given by the Corollary is the product  $\times p_n: \times \tilde{X}_n \rightarrow \times X_n$ , where  $p_n: \tilde{X}_n \rightarrow X_n$  is the covering projection  $\text{ex}: R \rightarrow S^1$ . The projection  $p$  is not a covering projection [1, p. 69], but it is a Hurewicz fibration (with unique path lifting).

**Remark 1.** Since every covering space of a connected locally path connected space is obtained from the path space  $PX$  by a construction similar to that in the Corollary, it follows a simple proof for the homotopy lifting property of a covering projection.

Now we give other examples of equivalence relations in the path space  $PX$  satisfying the conditions a) and b) and which do not correspond to any subgroup of the group  $\pi_1(X, x_0)$ , that is there is no subgroup  $H \subset \pi_1(X, x_0)$  such that  $\omega \rho \omega' \Leftrightarrow [\omega * \omega'^{-1}] \in H$ .

**Remark 2.** An equivalence relation  $\rho$  in  $PX$ , satisfying the conditions a) and b) corresponds to a subgroup of the group  $\pi_1(X, x_0)$  iff  $\omega \simeq \omega' \text{ rel } \dot{I} \Rightarrow \omega \rho \omega'$ . In fact, in this case the set  $\{[\omega]/\omega \rho \varepsilon_{x_0}\} = H$  is a subgroup of the group  $\pi_1(X, x_0)$ : if  $[\omega], [\omega'] \in H$ , then  $\omega \rho \varepsilon_{x_0}, \omega' \rho \varepsilon_{x_0}$  and from b) follows  $\omega * \omega' \rho \varepsilon_{x_0} * \omega'$  and the homotopy equivalence  $\varepsilon_{x_0} * \omega' \simeq \omega' \text{ rel } \dot{I}$  implies  $\omega * \omega' \rho \varepsilon_{x_0}$ , that is  $[\omega][\omega'] \in H$ . Then, if  $[\omega] \in H \Rightarrow \omega \rho \varepsilon_{x_0} \Rightarrow \omega * \omega^{-1} \rho \varepsilon_{x_0} * \omega^{-1} \Rightarrow \varepsilon_{x_0} \rho \omega^{-1} \Rightarrow [\omega^{-1}] \in H$ .

For this subgroup we have  $\omega \rho \omega' \Leftrightarrow [\omega * \omega'^{-1}] \in H$ .

**Example 2.** Consider in the space  $PX$  the equality relation. If the space  $X$  is non discrete, then this relation does not corresponds to any sub-



group of the group  $\pi_1(X, x_0)$ . In fact, the homotopy equivalence  $\omega \simeq \omega' \text{ rel } I$  implies  $\omega = \omega'$  only when  $X$  is a discrete space. However, the equality relation in  $PX$  gives the Serre fibration  $P: PX \rightarrow X$ .

*Example 3.* Let  $X = R^2$ ,  $x_0 = 0$  and let us consider the following equivalence relation in  $PR^2 = \{\omega: I \rightarrow R^2/\omega(0) = 0\}: \omega \rho \omega' \Leftrightarrow \omega(1) = \omega'(1)$  and  $\omega, \omega'$  has the same length. Then,  $\rho$  satisfies also the condition b) of the Theorem. We obtain the trivial fibration with the fibre  $R_1$ .

*Example 4.* Let  $G$  be a topological group with unity  $e$ . Let  $PG = \{\omega: I \rightarrow G/\omega \text{ continuous and } \omega(0) = e\}$ . We consider in  $PG$  the following equivalence relation:  $\omega \rho \omega' \Leftrightarrow \omega(1) = \omega'(1)$  and  $\omega \cdot \omega = \omega' \cdot \omega'$  where „ $\cdot$ ” is the multiplication in the group  $G$ .

The relation  $\rho$  satisfies the condition b) since we have the equality:  $(\omega * \theta)_{\frac{t+1}{2}} \cdot (\omega * \theta)_{\frac{t+1}{2}} = ((\omega \cdot \omega) * (\theta \cdot \theta))_{\frac{t+1}{2}}$ .

*Example 5.* Let  $p: E \rightarrow X$  be a Hurewicz fibration, corresponding to some equivalence relation  $\rho$  in  $PX = \{\omega \in X^I/\omega(0) = x_0\}$ , satisfying the conditions a) and b) of the Theorem.

Let  $Y$  be a path-connected space and let  $f: Y \rightarrow X$  be a continuous map. Let  $PY = \{\omega \in Y^I/\omega(0) = y_0 \in f^{-1}(x_0)\}$ . In  $PY$  we consider the following equivalence relation:  $w \rho' w' \Leftrightarrow w(1) = w'(1)$  and  $(fw) \rho (fw')$ . From the equality  $f(w * \theta)_{\frac{t+1}{2}} = (fw * f\theta)_{\frac{t+1}{2}}$ , it follows that the relation

$\rho'$  in  $PY$  satisfies the conditions a), b) of the Theorem. Thus we obtain a Hurewicz fibration  $p': E' \rightarrow Y$ . It is interesting to remark that this fibration does not coincide with the fibration induced by  $f$ . In fact, if we consider the constant map  $f(y) = x_0$ , then  $E'$  is homeomorphic to  $Y$ , but  $E_f = \{(e, y)/p(e) = f(y) = x_0\}$  is homeomorphic to  $Y \times F$ , where  $F$  is the fiber of  $p$  in the point  $x_0$ .

Also, if we consider the Serre fibration  $P: PX \rightarrow X$ ,  $Y$  a subspace of the space  $X$ ,  $x_0 \in Y$  and if  $i: Y \rightarrow X$  is the inclusion map, then  $E' = PY$ , but  $E_i = \{\omega \in PX/\omega(1) \in Y\}$ .

*Remark 3.* If the relation  $\rho$  of the Theorem is an open relation in  $PX$ , then the Hurewicz fibration  $p: E \rightarrow X$  is with unique path lifting. But in this case it can be proved that the relation is conjugate to the relation induced by the subgroup  $p_{\#}\pi_1(E, e_0)$  of the group  $\pi_1(X, x_0)$ , that is the equivalence relation  $\rho$  is induced by a subgroup of the group  $\pi_1(X, x_0)$ .

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### OBSERVAȚII ASUPRA FIBRĂRII DE DRUMURI A UNUI SPAȚIU TOPOLOGIC

(Rezumat)

Se generalizează construcția spațiilor de acoperire, obținând astfel unele fibrări Hurewicz. Dacă  $PX$  este spațiul drumurilor unui spațiu  $X$ , cu originea într-un punct dat  $x_0 \in X$ , atunci se consideră în  $PX$  relații de echivalență care satisfac condițiile a) și b). Se demonstrează atunci că spațiul factor obținut din  $PX$  prin această relație de echivalență este o fibrare Hurewicz cu baza  $X$ . Se dau o serie de exemple și se fac observații asupra acestei construcții.



## RECTANGULAR CONTINUITY

BY

GH. COSTOVICI

Let  $X_i, i \in I, \prod X_i$  be topological spaces (the last with the product topology),  $f: \prod X_i \rightarrow X$ , and  $k$  fixed in  $I$ . We note  $(x_i)_{i \in I - \{k\}}$  by  $(x_i; i \neq k)$  and closure = Cl. The function  $f$  is called  $k$ -continuous  $\Leftrightarrow$  for each  $(x_i; i \neq k), x_i \in X_i, i \in I - \{k\}$ , the mapping  $f_{(x_i; i \neq k)}: X_k \rightarrow X$  defined by  $f_{(x_i; i \neq k)}(x_k) = f(x_i; x_k = x_k)$  is continuous (in this definition it is not necessary that  $X_i, i \in I - \{k\}$  to be topological spaces). The function  $f$  is separately continuous if it is  $k$ -continuous for each  $k \in I$ . The function  $f$  is rectangular continuous  $\Leftrightarrow \forall Y_i \subseteq X_i, i \in I, f(\text{Cl } \prod Y_i) \subseteq \text{Cl } f(\prod Y_i)$ . If  $f$  is continuous then it is rectangular continuous. We remind that  $\text{Cl}(\prod Y_i) = \prod \text{Cl}(Y_i)$ .

**Proposition 1.** *If  $f$  is rectangular continuous then  $f$  is  $k$ -continuous for each  $k \in I$ .*

**Proof.** Let  $(x_i; i \neq k), x_i \in X_i, i \in I$  and  $A_k \subseteq X_k, k$  fixed in  $I$ . We have  $f_{(x_i; i \neq k)}(\text{Cl}(A_k)) = f(\prod x_i; x_k = \text{Cl}(A_k)) \subseteq f(\prod \text{Cl}(x_i); x_k = \text{Cl}(A_k)) = f(\text{Cl}(\prod x_i; x_k = A_k)) \subseteq \text{Cl}(f(\prod x_i; x_k = A_k)) = \text{Cl}(f_{(x_i; i \neq k)}(A_k))$ .

**Proposition 2.** *If  $I = \{1, 2, \dots, n\}$  and  $f$  is  $k$ -continuous for every  $k \in I$ , then  $f$  is rectangular continuous.*

**Proof.** Always we have

$$(\alpha) \quad \bigcup_Y \text{Cl}(E_Y) \subseteq \text{Cl}\left(\bigcup_Y E_Y\right).$$

Let  $A_i \subseteq X_i, i \in I$  where  $A_i = \text{Cl}(B_i), i$  fixed in  $I$ . We have

$$\begin{aligned} f(\pi A_i) &= \bigcup_{(a_i \in A_i; i \neq l)} f(\pi a_i; a_l = \text{Cl}(B_l)) = \bigcup_{(a_i \in A_i; i \neq l)} f_{(a_i \in A_i; i \neq l)} \text{Cl}(B_l) \subseteq \\ &\subseteq \bigcup_{(a_i \in A_i; i \neq l)} \text{Cl}(f_{(a_i \in A_i; i \neq l)}(B_l)) = \bigcup_{(a_i \in A_i; i \neq l)} \text{Cl}(f(\Pi a_i; a_l = B_l)) \subseteq \\ &\subseteq \text{Cl}\left(\bigcup_{(a_i \in A_i; i \neq l)} f(\Pi a_i; a_l = B_l)\right) = \text{Cl } f(\Pi A_i; A_l = B_l). \end{aligned}$$

In the last but one inclusion we have used the fact that  $f$  is  $l$ -continuous, and in the last we have used  $(\alpha)$ . We have obtained

$$(\beta) \quad f(\Pi A_i; A_l = \text{Cl}(B_l)) \subseteq \text{Cl}(f(\Pi A_i; A_l = B_l)).$$

We note that  $(\beta)$  is true even if  $I$  is infinite.

Now, if  $Y_i \subseteq X_i, i \in I$ , applying  $(\beta)$  of  $n$  times, we have,  $f(\text{Cl}(\Pi Y_i)) = f(\Pi \text{Cl}(Y_i)) = f(\text{Cl}((Y_1) \times \text{Cl}(Y_2) \times \dots \times \text{Cl}(Y_n))) \subseteq \text{Cl}(f(Y_1 \times \text{Cl}(Y_2)) \times \dots \times \text{Cl}(Y_n)) \subseteq \text{Cl} \text{Cl}(f(Y_1 \times Y_2 \times \text{Cl}(Y_3)) \times \dots \times \text{Cl}(Y_n)) \subseteq \dots \subseteq \text{Cl} \text{Cl} \dots \text{Cl} f(\Pi Y_i) = \text{Cl}(f(\Pi Y_i))$ .

*Special case.* If  $G$  is multiplicative grupoid equipped with a topology, we say that the topology is left compatible (right compatible) (compatible)  $\Leftrightarrow$  the operation is right continuous (left continuous) (separately continuous)  $=$  rectangular continuous) respectively  $\Leftrightarrow \forall x \in G$  and  $\forall A, B \subseteq G$ , we have  $x \text{Cl}(A) \subseteq (\text{Cl}(xA) \subseteq (\text{Cl} A)x \subseteq (\text{Cl}(Ax))(\text{Cl}(A)\text{Cl}(B) \subseteq \text{Cl}(AB)) \subseteq$  respectively.

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### CONTINUITATE DREPTUNGHILARĂ

(Rezumat)

O funcție  $f$  definită pe un produs topologic este dreptunghiular continuă dacă ea satisface  $f(\Pi \bar{A}_i) \subseteq \bar{f}(\Pi A_i)$ . Se arată că dacă  $f$  este dreptunghiular continuă atunci ea este separat continuă și reciproc, dacă produsul este finit și  $f$  este separat continuă atunci  $f$  este dreptunghiular continuă.



## A GENERAL MATHEMATICAL NOTION OF THE ALGORITHM

BY

MARIA CAZACU

**1. Preliminaries.** In the present paper we introduce a general mathematical notion of the algorithm and several particular classes of algorithms are then underlined.

Let  $A$  and  $B$  be sets. A *binary relation* from  $A$  to  $B$  is any subset  $\alpha \subseteq A \times B$ . The notation  $a \alpha b$  means (that  $a, b \in \alpha$ ).

The *domain* and the *range* of  $\alpha$  are the sets

$$\text{Dom}(\alpha) = \{a \mid \exists b : a \alpha b\} \subseteq A,$$

$$\text{Ran}(\alpha) = \{b \mid \exists a : a \alpha b\} \subseteq B.$$

They say that relation  $\alpha$  is *total defined* if  $\text{Dom}(\alpha) = A$ . Let  $a \in A$ . If  $\alpha(a) \neq \emptyset$  they say that relation  $\alpha$  is defined for  $a$  and we write it  $\alpha(a) \downarrow$ . If  $\alpha(a) = \emptyset$  they say that relation  $\alpha$  is undefined for  $a$  and we write  $\alpha(a) \uparrow$ .

We say that relation  $\alpha$  is of *type*  $(m, 1)$  if

$$a \alpha b_1 \text{ and } a \alpha b_2 \Rightarrow b_1 = b_2,$$

of *type*  $(1, m)$  if

$$a_1 \alpha b \text{ and } a_2 \alpha b \Rightarrow a_1 = a_2$$

and of *type*  $(1, 1)$  or *injective* if it is of type  $(m, 1)$  and of type  $(1, m)$  at the same time. The relations of type  $(m, 1)$  are called functions or applications too.

We shall name *code* or *representation* of the set  $A$  in the set  $B$  any total defined relation  $\alpha : A \rightarrow B$  of type  $(1, m)$ . The notion of code allows the introduction of simulation relation between the functions.

Let  $f : X_1 \rightarrow Y_1$ ,  $g : X_2 \rightarrow Y_2$  be functions and let  $\alpha : X_1 \rightarrow X_2$ ,  $\gamma : Y_1 \rightarrow Y_2$  be codes. We shall say that the function  $f$  is  $(\alpha, \gamma)$ -*simulated* by the function  $g$  (notation:  $f \leq g(\alpha, \gamma)$ ) if

$$\alpha g \subseteq f \gamma \text{ and } \alpha^{-1} f \subseteq g \gamma^{-1}.$$

The function  $f$  is *simulated* by the function  $g$  (notation:  $f \leq g$ ) if there are the codes  $\alpha$  and  $\gamma$  so that  $f \leq g(\alpha, \gamma)$ . The simulation relation between the functions is reflective and transitive.

Let  $D$  be an infinite countable fixed set whose elements we shall name *decisions* and  $S$  a set whose elements we shall name *states*. By the



statement on  $S$  we shall understand any application  $I: S \rightarrow S \times D$  with the property that the set

$$D(I) = \{d \mid \exists s_1, s_2 : I(s_1) = (s_2, d)\}$$

is finite.  $D(I)$  will be called the set of decisions of the statement  $I$ . To every statement  $I$  is associated a transformation function  $I^\tau: S \rightarrow S$  and a decision function  $I^d: S \rightarrow D$  as follows. Let  $s \in S$ . If  $I(s) = (s_1, d)$ , then  $I^\tau(s) = s_1$  and  $I^d(s) = d$ ; if  $I(s) \uparrow$  then  $I^\tau(s) \uparrow$  and  $I^d(s) \uparrow$ .

Instructions  $d_a$  with the property  $d_a(s) = (s, a)$  for any  $s \in S$  will be called decisional statements. We shall note with  $\Delta$  the set of these statements.

Let  $J$  be a set of statements on the set of states  $S$ . By algorithm on  $I$  we shall understand any system  $A = (X, U, x_1, \omega, \beta)$  where  $X$  is a finite set of nodes (addresses),  $U \subset X \times X$  is a set of edges,  $x_1 \in X$  is the initial node,  $\omega: U \rightarrow PF(D)$  (with  $PF(D)$  we are noted the set of finite parts of  $D$ ) is a total defined application named succession function,  $\beta: X \rightarrow I \cup \Delta$  is a total defined application named concretization function of the nodes. The statements of the set  $\beta(X)$  are the components of  $A$ . By  $ALG(J)$  we shall denote the set of all algorithms on  $I$ .

Let  $A$  be an algorithm on  $J$  and  $S \times (X \cup D)$  the set of configurations. The configurations of the form  $(s, x) \in S \times X$  are nonfinals and those of the form  $(s, d) \in S \times D$  are finals. Further

$$(s, x) \xrightarrow{A} (s_1, y) \text{ and } (s, x) \xrightarrow{A} . (s_1, d)$$

mean

$$\beta_x(s) = (s_1, d) \wedge d \in \omega(x, y) \text{ and } \beta_x(s) = (s_1, d) \wedge \neg \exists y (d \in \omega(x, y))$$

respectively.

These are elementary transformation or steps in the work of  $A$ .

Every algorithm  $A$  defines a statement  $\bar{A}$  on  $S$  as follows. Let  $s \in S$ . If there is a finite string of configurations  $(s_1, y_1), (s_2, y_2), \dots, (s_n, y_n), (s', d)$  so that

$$(s_1, x_1) \xrightarrow{A} (s_1, y_1) \xrightarrow{A} (s_2, y_2) \xrightarrow{A} \dots \xrightarrow{A} (s_n, y_n) \xrightarrow{A} . (s', d)$$

then  $\bar{A}(s) = (s', d)$ . Otherwise,  $\bar{A}(s) \uparrow$ .

Two algorithms  $A$  and  $B$  are equivalents if  $\bar{A} = \bar{B}$ .

An algorithm may be represented as a finite-oriented graph or as program. A program of an algorithm  $A = (X, U, x_1, \omega, \beta)$  having  $p$  nodes is a string of  $p$  expressions (one expression for each node) of the form

$$x: I, D_1/y_1, D_2/y_2, \dots, D_n/y_n, D'/\downarrow;$$

where  $x$  is a node,  $I = \beta(x)$  and  $D_i/y_i$  means that if  $I$  takes a decision  $d \in D_i$  then follows the statement in the node  $y_i$  and if  $I$  takes a decision of  $D'$  then  $A$  stops with the respective decision. At last, if  $I$  takes a decision  $d \notin D_1 \cup D_2 \cup \dots \cup D_n \cup D'$  then follows the statement of the following expression if there exists any, otherwise the program stops with this decision. First expression of program is for initial node. Each of the parts  $D_1/y_1, \dots, D_n/y_n$  or  $D'/\downarrow$  may be absent. Sometimes we shall use the following simpli-



fication conventions. If there isn't in the program an "appeal"  $D_k/x$  to the expression  $x:I, \dots$  then " $x$ :" will be eliminated. The *product*  $IJ$ , *d-product*  $IdJ$  and *d-iteration*  $I^d$  are statements defined by the algorithms

$I; J; I, D(I) - \{d\}/\downarrow; J;$  and  $1:I d/1; I$ , respectively.

*Example.* Let be the algorithm in the Fig. 1 in which we suppose that the decision sets of the statements  $I_1, I_2, I_3, I_4$  coincide with the set  $\{0, 1, 2\}$ . This algorithm will have the following representation as the form of a program:

$x:I_1, 1, 2/z; I_2, 1/w, 0, 2; z:I_3, 0, 1, 2/z; w:I_4, 1/z, 2/x;$

The different algorithms taken into account will be represented as the form of a program.

**2. Examples of Classes of Algorithms.** Further we shall give some examples of classes (sets) of algorithms. We suppose that in them the set of decisions  $D$  is the set of natural numbers.

a) *Turing algorithms.* Let  $A = \{a_0, a_1, \dots, a_{n-1}\}$  be an unvoid finite set called alphabet,  $* \notin A$  and let  $S$  be the set of states of the forms  $s = \dots b_2 b_1 b_0 * c_0 c_1 c_2 \dots$  where  $b_i, c_i \in A$  for  $i = 0, 1, 2$  and all but finitely many  $b_i, c_i$  are  $a_0$ , which we are called the *repose symbol*. Let us consider the statements  $L, R, T$  on  $S$  defined for any  $u, v \in A^*$  and  $a_i, b_i, c \in A$  as follows:

$$L(U a_i * bcV) = (U * a_i bcV, i),$$

$$R(Ub * ca_i V) = (Ubc * a_i V, i),$$

$$T(Ub * a_i V) = (Ub * a_j V, j)$$

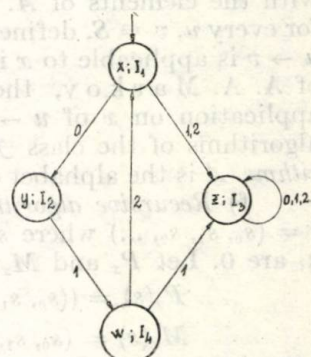


Fig. 1

where  $j = i + 1$  if  $0 \leq i < n - 1$  and  $j = 0$  if  $i = n - 1$ .

The algorithms of the class  $T_n = \text{ALG}(\{L, R, T\})$  are called *Turing algorithms*. It is evident the similarity with the Turing machines.

b) *Turing algorithms of order k.* Let  $A$  be the above-mentioned alphabet. Now, a state is every  $k$ -tuples ( $k \geq 1$ )  $s = (s_1, s_2, \dots, s_k)$  where  $s_i = \dots b_2^i b_1^i b_0^i * c_0^i c_1^i c_2^i \dots$  so that  $b_j^i, c_j^i \in A$  and all but finitely many  $b_j^i, c_j^i$  are  $a_0$ . Let  $S$  be the set of these states and let  $L_i, R_i, T_i$  ( $i = 1, 2, \dots, k$ ) be the statements on  $S$  whose action over one state  $s = (s_1, s_2, \dots, s_i, \dots, s_k)$  is reduced at one action over  $s_i$  similarly with the action of the Turing statements  $L, R, T$  respectively. Let  $T_n^k = \text{ALG}(\{L_1, \dots, L_k, R_1, \dots, R_k, T_1, \dots, T_k\})$ .

The algorithms of the class  $T_n^k$  are called *Turing algorithms of order k*, with the alphabet of the external states formed from  $n$  elements. Obviously, the class  $T_n^1$  coincides with the above defined class  $T_n$ .

c) *The class  $L_n^k$ .* Let  $S$  be the set of states of the form (1)  $s = (s_1, s_2, \dots, s_k)$ , where  $s_i = b_0^i b_1^i \dots b_p^i * c_0^i c_1^i c_2^i, \dots, b_j^i, c_j^i \in A$  and almost all  $c_0^i, c_1^i, \dots$  are equal to the repose symbol  $a_0$ . The statements  $R_i, T_i$  are defined as in the above mentioned case, taking into account the new form states, infinite only to the right. The  $L_i$  statement transforms a state  $s$  of the form (1) as follows. It always leaves unchanged the strings  $s_1, s_2, \dots, s_{i-1}, s_{i+1}, \dots, s_k$ , leaves unchanged the string  $s_i$  if this begins with the symbol \*



and otherwise transforms  $s_i$  in the string  $b_0^i b_1^i \dots b_{p-1}^i * b_p^i c_0^i c_1^i \dots$ . The decision taken by  $L_i$  will be  $j$  if  $a_j$  is the symbol situated on the very right of the symbol  $*$  in  $s_i$  after the above mentioned transformations. Then  $L_n^k = \text{ALG}(\{L_1, \dots, L_k, R_1, \dots, R_k, T_1, \dots, T_k\})$ . These algorithms are similar to the Turing machines with  $k$  limited on left tapes and  $n$  symbols on the tape.

d) *Operatorial algorithms*. Let  $S = \{2^a 3^b 5^c \mid a, b, c \in N\}$  and the statements  $q_2, q_3, q_5$  and  $r$  defined as

$$q_i(x) = (i \cdot x, 1) \text{ for } i = 2, 3, 5 \text{ and}$$

$$r(x) = \begin{cases} \left(\frac{x}{30}, 1\right) & \text{if } 30 \mid x, \\ (x, 0) & \text{if } 30 \nmid x. \end{cases}$$

The algorithms of the class  $O = \text{ALG}(\{q_2, q_3, q_5, r\})$  are *operatorial algorithms*.

e) *Markov algorithms*. Let  $S = A^*$  be the set of all finite strings with the elements of  $A$ . Let us consider the statements of the form  $T_{u,v}$  for every  $u, v \in S$ , defined for any word  $x \in S$  as follows. If the substitution  $u \rightarrow v$  is applicable to  $x$  in the sense of the theory of the normal algorithms of A. A. Markov, then  $T_{u,v}(x) = (x', 1)$  where  $x'$  is the result of the application on  $x$  of  $u \rightarrow v$  substitution. Otherwise,  $T_{u,v}(x) = (x, 0)$ . The algorithms of the class  $\mathcal{M}_n = \text{ALG}(\{T_{u,v} \mid u, v \in A^*\})$  are *Markov algorithms*.  $A$  is the alphabet of Markov algorithms from the class  $\mathcal{M}_n$ .

f) *Recursive algorithms*. Let  $S$  be the set of states having the form  $s = (s_0, s_1, s_2, \dots)$  where  $s_i$  are natural numbers and all but finitely many  $a_i$  are 0. Let  $P_x$  and  $M_x$  ( $x = 0, 1, 2, \dots$ ) the algorithms on  $S$  defined by

$$P_x(s) = ((s_0, s_1, \dots, s_{x-1}, s_x + 1, s_{x+1}, \dots), 1),$$

$$M_x(s) = (s_0, s_1, \dots, s_{x-1}, s_x \div 1, s_{x+1}, \dots) \text{ sg}(s_x - 1),$$

where  $0 \div 1 = 0$ ,  $(a + 1) \div 1 = a$ ,  $\text{sg}(0) = 0$  and  $\text{sg}(a + 1) = 1$  for all natural numbers  $a$ . The recursive algorithms are the algorithms of the set  $\mathcal{R} = \text{ALG}(\{P_0, M_0, P_1, M_1, P_2, M_2, \dots\})$ .

Let  $J$  be a set of statements. We shall denote by  $\bar{J}$  the set of statements defined by all algorithms with the components from  $J$ .

The set  $J$  is *closed* if  $\bar{J} = J$ . Let  $J$  be a closed set. The subset  $J \subset J$  is *complete system* in  $J$  if  $\bar{J} = J$ . The complete system  $J$  is called *base* in  $J$  if any proper part of  $J$  is no more complete system in  $J$ .

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## O NOȚIUNE MATEMATICĂ GENERALĂ DE ALGORITM

(Rezumat)

Se consideră o noțiune matematică generală de algoritm și se introduce în calitate de clase particulare algoritmii Turing de diverse tipuri, algoritmii operatoriali, recursivi și Markov.



# UNIFORM ASYMPTOTIC STABILITY FOR SYSTEMS OF VOLTERRA INTEGRAL EQUATIONS

BY

ADRIAN CORDUNEANU

Consider the integral system

$$(1) \quad u(t) = \varphi(t) + \int_{t_0}^t K(t, s, u(s)) ds, \quad t \geq t_0$$

where  $u$ ,  $\varphi$  and  $K$  are  $n$ -vectors,  $t$  and  $s$  are real variables,  $t_0 \geq a$ ,  $a$  being a real number, fixed in all what follows. The kernel  $K = K(t, s, u)$  is assumed defined and continuous for  $a \leq s \leq t < +\infty$ ,  $|u| < b$  where  $|\cdot|$  is an euclidean norm in  $R^n$ . We also suppose that for every continuous  $\varphi = \varphi(t)$ , the system (1) has unique solution  $u = u(t)$ , defined for  $t \geq t_0$ .

In this paper, we assume that  $\varphi = \varphi(t)$  belongs to a normed space  $N$  of continuously differentiable functions, defined for  $t \geq a$ , with the norm  $\|\cdot\|$ . Denote by  $\mathcal{C}$  the space of all constant vector-functions, with the norm  $\|\varphi\| = |\varphi|$ . Denote by  $\mathcal{C}_1[a, +\infty)$  the space of all continuously differentiable vector-functions  $\varphi = \varphi(t)$ , such that  $\varphi(t)$  and  $\varphi'(t)$  are bounded for  $t \geq a$ . The norm in this space is given by

$$(2) \quad \|\varphi\| = \max \left\{ \sup_{t \geq a} |\varphi(t)|, \sup_{t \geq a} |\varphi'(t)| \right\}.$$

Finally, denote by  $\mathcal{C}_1^p[a, +\infty)$  the space of all continuous bounded for  $t \geq a$  vector-functions  $\varphi = \varphi(t)$ , for which  $\varphi'(t)$  is continuous and

$\int_a^\infty |\varphi'(t)|^p dt < +\infty$ , where  $1 < p < +\infty$ . The norm in this last space is given by

$$(3) \quad \|\varphi\| = \left( \sup_{t \geq a} |\varphi(t)| \right) + \left( \int_a^\infty |\varphi'(t)|^p dt \right)^{1/p}.$$

In [1] are defined and studied three types of stability for the solution of the system (1). We say, following J.M. B o w n d s and J.M. C u s h i n g, that the solution  $u = \bar{u}(t)$  of (1) corresponding to  $\varphi = \bar{\varphi}(t) \in N$  is stable

on  $N$ , if for every  $\varepsilon > 0$  and  $t_0 \geq a$  there exists  $\delta(\varepsilon, t_0) > 0$ , such that if  $\|\varphi - \bar{\varphi}\| < \delta(\varepsilon, t_0)$  then  $|u(t) - \bar{u}(t)| < \varepsilon$  for  $t \geq t_0$ . If  $\delta = \delta(\varepsilon) > 0$  may be found independently of  $t_0 \geq a$ , the solution  $u = \bar{u}(t)$  is called uniformly stable (U.S.) on  $N$ . The solution  $u = \bar{u}(t)$  of (1) is called asymptotically stable (A.S.) on  $N$  if it is stable on  $N$  and if, in addition, for every  $t_0 \geq a$  there exists a number  $r(t_0) > 0$ , such that  $\|\varphi - \bar{\varphi}\| < r(t_0)$  implies  $|u(t) - \bar{u}(t)| \rightarrow 0$  as  $t \rightarrow +\infty$ .

**Remark 1.** In the scalar case  $n = 1$ , the stability on  $\mathcal{C}$  of the solution  $\bar{u}(t) \equiv 0$  of the linear equation

$$(4) \quad u(t) = \varphi(t) + \int_{t_0}^t a(t, s)u(s)ds, \quad t \geq t_0,$$

where  $a(t, s) \geq 0$ , is equivalent with the stability on the space  $\mathcal{C}[a, +\infty)$ , consisting of all continuous bounded vector functions defined for  $t \geq a$ , with the norm

$$(5) \quad \|\varphi\| = \sup_{t \geq a} |\varphi(t)|, \quad \varphi \in \mathcal{C}[a, +\infty).$$

This follows very easy, if we take into account [2, p. 93] that from the inequalities

$$(6) \quad u(t) \leq \|\varphi\| + \int_{t_0}^t a(t, s)u(s)ds, \quad t \geq t_0$$

and

$$(7) \quad u(t) \geq -\|\varphi\| + \int_{t_0}^t a(t, s)u(s)ds, \quad t \geq t_0$$

we obtain  $w_1(t) \leq u(t) \leq w_2(t)$  for  $t \geq t_0$ , where

$$(8) \quad w_1(t) = -\|\varphi\| + \int_{t_0}^t a(t, s)w_1(s)ds, \quad t \geq t_0$$

and respectively

$$(9) \quad w_2(t) = \|\varphi\| + \int_{t_0}^t a(t, s)w_2(s)ds, \quad t \geq t_0.$$

The aim of this Note is to define a new concept of stability for the solution of (1).

**Definition.** We say that the solution  $u = \bar{u}(t)$  of the system (1), corresponding to  $\varphi = \bar{\varphi}(t) \in N$  is uniformly asymptotically stable (U.A.S.) on  $N$ , if



(a) this solution is uniformly stable (U.S.) on  $N$ ;

(b) there exists  $r > 0$ , such that for every  $\eta > 0$  we find  $T(\eta) > 0$  having the property that

$$(10) \quad |u(t) - \bar{u}(t)| < \eta \quad \text{if } t \geq t_0 + T(\eta), \quad \|\varphi - \bar{\varphi}\| < r$$

independently of  $t_0 \geq a$ .

*Remark 2.* The above condition (b) means that  $|u(t) - \bar{u}(t)| \rightarrow 0$  when  $t - t_0 \rightarrow +\infty$ , uniformly with respect to  $\varphi$  satisfying the inequality  $\|\varphi - \bar{\varphi}\| < r$ .

*Remark 3.* If  $\bar{u}(t) \equiv 0$  is a solution of (1), we say that the system (1) is stable (U.S., A.S. or U.A.S.) on  $N$ , provided that this solution is stable (U.S., A.S. or U.A.S.) on  $N$ .

Remember [1, p. 68] that if  $U = U(s, t)$  is the solution of the matrix equation

$$(11) \quad U(s, t) = I + \int_s^t A(t, \tau) U(s, \tau) d\tau, \quad a \leq s \leq t < +\infty,$$

where  $I$  is the identity  $n \times n$  matrix and  $A = A(t, \tau)$  is a  $n \times n$  continuous matrix for  $a \leq \tau \leq t < +\infty$ , then the vector-function

$$(12) \quad u(t) = U(t_0, t)\varphi(t_0) + \int_{t_0}^t U(s, t) \left[ \varphi'(s) + \frac{d}{ds} \left( \int_a^s B(s, \tau) d\tau \right) \right] ds, \quad t \geq t_0$$

solves the integral equation

$$(13) \quad u(t) = \varphi(t) + \int_{t_0}^t [A(t, s)u(s) + B(t, s)] ds, \quad t \geq t_0,$$

provided  $B(t, s)$  is a continuous vector-function, possessing a continuous derivative in  $t$  on  $t_0 \leq s \leq t < +\infty$  and  $\varphi = \varphi(t)$  is a continuous differentiable on  $[t_0, +\infty)$  vector-function.

In this paper, we define the matrix norm of a matrix  $A$  to be  $|A| = \sup |A\xi|$ , where  $\xi$  is a  $n$ -vector,  $|\xi| = 1$ .

**Proposition 1.** If  $U = U(s, t)$  solves the equation (11), then the linear system

$$(14) \quad u(t) = \varphi(t) + \int_{t_0}^t A(t, s)u(s) ds, \quad t \geq t_0$$

is U.A.S. on  $\mathcal{C}$  if and only if

(i) there exists a constant  $M > 0$  independent of  $t_0$ , such that  $|U(t_0, t)| \leq M$  for  $t \geq t_0 \geq a$ ;

(ii) for every  $\eta > 0$  we find  $T(\eta) > 0$  having the property that  $|U(t_0, t)| < \eta$  if  $t \geq t_0 + T(\eta)$ , independently of  $t_0 \geq a$ .

**Proof.** Because in the case  $\varphi \in \mathcal{O}$ , the solution of (14) is given by  $u(t) = U(t_0, t) \varphi(t_0) = U(t_0, t) \varphi$ , it suffices to prove only that if (14) is U.A.S. on  $\mathcal{O}$ , then (i) + (ii) are satisfied. The condition (i) being equivalent with the fact that (14) is U.S. on  $\mathcal{O}$ , it remains to justify (ii). Taking  $\eta' > 0$  and  $t_0 \geq a$ , we have  $|u(t)| = |U(t_0, t) \varphi| < \eta'$  if  $t \geq t_0 + T_1(\eta')$  and  $\|\varphi\| = |\varphi| < r$ , where  $r > 0$  is a fixed number. Taking  $\varphi \in \mathcal{O}$  with  $|\varphi| = r/2$ , it follows  $|U(t_0, t) \varphi| < \eta'$  for  $t \geq t_0 + T_1(\eta')$ , which implies  $|U(t_0, t)| \leq 2\eta' r^{-1}$  for  $t \geq t_0 + T_1(\eta')$ . Finally, putting  $\eta' = \eta r/4$  and denoting  $T_1(\eta r/4)$  by  $T(\eta)$ , we have  $|U(t_0, t)| < \eta$  for  $t \geq t_0 + T(\eta)$  which concludes the proof.

**Proposition 2.** Under the conditions (i), (ii) and (iii)

$\lim_{\uparrow T_\infty} \int_T^t |U(s, t)| ds = 0$ , uniformly with respect to  $t \geq T$ , the linear system

(14) is U.A.S. on  $\mathcal{O}_1[a, +\infty)$ .

**Proof.** We can write for the solution  $u = u(t)$  of the system (14) the relation

$$(15) \quad u(t) = U(t_0, t) \varphi(t_0) + \int_{t_0}^t U(s, t) \varphi'(s) ds, \quad t \geq t_0,$$

from which we get

$$(16) \quad |u(t)| \leq |U(t_0, t)| |\varphi(t_0)| + \int_{t_0}^t |U(s, t)| |\varphi'(s)| ds, \quad t \geq t_0.$$

Taking into account (i) and (ii), we obtain

$$(17) \quad |u(t)| \leq \|\varphi\| \left( M + \int_{t_0}^t |U(s, t)| ds \right), \quad t \geq t_0.$$

Using the condition (iii), we conclude that there exists a number  $T_1 > a$  such that

$$(18) \quad \int_{T_1}^t |U(s, t)| ds < 1 \quad \text{for all } t \geq T_1.$$

If  $t_0 < T_1$ , it follows for  $t \geq T_1$

$$(19) \quad \int_{t_0}^t |U(s, t)| ds = \int_{t_0}^{T_1} |U(s, t)| ds + \int_{T_1}^t |U(s, t)| ds \leq M_1 + 1,$$

where by  $M_1$  we have denoted the upper bound of the values of the integral



$\int_a^{T_1} |U(s, t)| ds$  on  $[T_1, +\infty)$ . This number is finite, because by (ii) for a fixed  $s$  it follows  $|U(s, t)| \rightarrow 0$  when  $t \rightarrow +\infty$  and thus

$$(20) \quad \lim_{t \rightarrow \infty} \int_a^{T_1} |U(s, t)| ds = 0.$$

On the other hand, for  $t_0 \leq t \leq T_1$ , we have

$$(21) \quad \int_{t_0}^t |U(s, t)| ds \leq \int_a^{T_1} |U(s, t)| ds \leq M_2,$$

where  $M_2$  is a constant. Putting  $M_3 = \max \{M_1 + 1, M_2\}$ , we have in the case  $t_0 < T_1$ , that

$$(22) \quad \int_{t_0}^t |U(s, t)| ds \leq M_3 \quad \text{for all } t \geq t_0.$$

Turning back to (17), we obtain for the solution of the system (14) the following majoration

$$(23) \quad |u(t)| \leq (M + M_3) \|\varphi\|, \quad t \geq t_0.$$

If  $t_0 \geq T_1$ , again using (iii), it follows

$$(24) \quad \int_{t_0}^t |U(s, t)| ds < 1 \quad \text{for } t \geq t_0,$$

consequently (23), where  $M$  and  $M_3$  are constant, is true independently of  $t_0 \geq a$ . This proves that the condition (a) from the U.A.S. definition is satisfied. Imposing the restriction  $\|\varphi\| < 1$ , we shall obtain from (16) the following inequality

$$(25) \quad |u(t)| \leq |U(t_0, t)| + \int_{t_0}^t |U(s, t)| ds, \quad t \geq t_0.$$

Taking  $\eta > 0$ , we remark that accordingly with (ii) it follows  $|U(t_0, t)| < \eta/3$  for  $t \geq t_0 + T_2(\eta)$ . Using (iii) we find  $T_3 = T_3(\eta)$  such that

$$(26) \quad \int_{T_2}^t |U(s, t)| ds < \frac{\eta}{3} \quad \text{if } t \geq T_3 = T_3(\eta).$$

If  $t_0 < T_3(\eta)$ , we can write for  $t \geq T_3(\eta)$

$$(27) \quad \int_{t_0}^t |U(s, t)| ds = \int_{t_0}^{T_2} |U(s, t)| ds + \int_{T_2}^t |U(s, t)| ds.$$

Since

$$(28) \quad |U(s, t)| < \frac{\eta}{3 T_3(\eta)} \quad \text{for} \quad t \geq T_4(\eta), \quad s \leq T_3(\eta)$$

in the case  $t_0 < T_3(\eta)$ , we obtain finally

$$(29) \quad |u(t)| < \eta \quad \text{for} \quad t \geq t_0 + T(\eta), \quad \|\varphi\| < 1,$$

where  $T(\eta)$  is given by

$$(30) \quad T(\eta) = \max \{T_2(\eta), T_3(\eta) - a, T_4(\eta) - a\}.$$

If  $t_0 \geq T_3(\eta)$ , the inequality (29) also holds, where  $T(\eta)$  is given again by (30). Thus the condition (b) from the U.A.S. definition is also satisfied. This concludes the proof.

In a similar way, may be proved also the following

**Proposition 3.** Under the conditions (i), (ii) and (iv)

$$\lim_{T \rightarrow \infty} \int_T^t |U(s, t)|^q ds = 0, \text{ uniformly with respect to } t \geq T \text{ where } \frac{1}{p} + \frac{1}{q} = 1,$$

the linear system (14) is U.A.S. on  $\mathcal{C}_1^p[a, +\infty)$ .

For the study of U.A.S. of the linear perturbed system

$$(31) \quad u(t) = \varphi(t) + \int_{t_0}^t [A(t, s)u(s) + f(t, s, u(s))] ds, \quad t \geq t_0,$$

the following hypotheses will be used :

(H<sub>1</sub>)  $f(t, s, u)$  is a continuous vector-function for  $a \leq s \leq t < +\infty$ ,  $|u| < b$ ,  $|f(t, s, u)| \leq \gamma_1(t) |u|$  where  $\gamma_1(t)$  is a continuous scalar function

$$\text{and} \quad \int_a^\infty \gamma_1(t) dt < +\infty;$$

(H<sub>2</sub>)  $f$  is continuously differentiable in  $t$ ,  $|f_t(t, s, u)| \leq \gamma_2(t, s) |u|$  for  $a \leq s \leq t < +\infty$ ,  $|u| < b$  where  $\gamma_2(t, s)$  is a continuous scalar function

$$\text{and} \quad \int_a^\infty dt \int_a^t \gamma_2(t, s) ds < +\infty.$$

**Proposition 4.** Under the conditions that the linear system (14) is U.S. on  $\mathcal{C}$ , U.A.S. on the normed space  $N$  and (H<sub>1</sub>) + (H<sub>2</sub>) are satisfied, the linear perturbed system (31) is U.A.S. on  $N$ .



**P r o o f.** Denote by

$$(32) \quad \Phi(t_0, t) = U(t_0, t)\varphi(t_0) + \int_{t_0}^t U(s, t)\varphi'(s) ds, \quad t \geq t_0,$$

the solution of (14) for  $\varphi \in N$ ; also denote

$$(33) \quad \gamma(t) = \gamma_1(t) + \int_{t_0}^t \gamma_2(t, s) ds, \quad v(t) = \sup_{t_0 \leq s \leq t} |u(s)| \text{ for } t \geq 0.$$

Taking into account that the solution  $u = u(t)$  of (31) satisfies the integral equation

$$(34) \quad u(t) = \Phi(t_0, t) + \int_{t_0}^t U(s, t) \left( \frac{d}{ds} \int_{t_0}^s f(s, \tau, u(\tau)) d\tau \right) ds, \quad t \geq t_0,$$

and using the Gronwall inequality together with the hypotheses  $(H_1) + (H_2)$ , we obtain as in [1, p. 71] that

$$(35) \quad v(t) \leq L \left( \sup_{t^* \geq t_0} |\Phi(t_0, t^*)| \right), \quad t \geq t_0,$$

where  $L > 0$  is a constant, given by

$$(36) \quad L = \exp \left( M \int_{t_0}^{\infty} \gamma(t) dt \right).$$

Here  $M$  is a constant for which, accordingly to U.S. of the system (14) on  $\mathcal{Q}$ , we have  $|U(s, t)| \leq M$  on  $a \leq s \leq t < +\infty$ . From (35) it follows that the system (31) is U.S. on  $N$ , because (14) has this property. From (35) also holds that for  $\varphi \in N$  with  $\|\varphi\|$  sufficiently small, we have  $|u(t)| \leq c$  for  $t \geq t_0$ , where  $c$  is some positive constant. Using the representation formula (34), we obtain

$$(37) \quad |u(t)| \leq |\Phi(t_0, t)| + c \int_{t_0}^t |U(s, t)| \gamma(s) ds, \quad t \geq t_0.$$

Taking  $\eta > 0$ , we have  $|\Phi(t_0, t)| < \eta/3$  for  $t \geq t_0 + T_1(\eta)$  and  $\varphi \in N$  with  $\|\varphi\| < r = \text{fixed number}$ , since (14) is U.A.S. on  $N$ . Let  $T_2 = T_2(\eta)$

be a number for which  $\int_{T_1}^{\infty} \gamma(s) ds < \frac{1}{3} \eta c M$ . This insures that for  $t \geq T_2$

we can write

$$(38) \quad c \int_{T_1}^t |U(s, t)| \gamma(s) ds \leq c M \int_{T_1}^{\infty} \gamma(s) ds < \frac{\eta}{3}.$$

If  $t_0 < T_2(\eta)$ , it follows

$$(39) \quad c \int_{t_0}^{T_2} |U(s, t)| \gamma(s) ds < \frac{\eta}{3}, \quad \text{for } t \geq T_3(\eta)$$

using the condition (ii) in Proposition 1. Consequently, if  $t_0 < T_2(\eta)$ , we have  $|u(t)| < \eta$  for  $t \geq t_0 + T(\eta)$  and  $\varphi \in N$  with  $\|\varphi\| < r$ , where  $T(\eta)$  is given by

$$(40) \quad T(\eta) = \max \{ T_1(\eta), T_2(\eta) - a, T_3(\eta) - a \}.$$

If  $t_0 \geq T_2(\eta)$ , from (37) we find  $|u(t)| < 2\eta/3$  for  $t \geq t_0 + T(\eta)$ ,  $\varphi \in N$  with  $\|\varphi\| < r$ , where  $T(\eta)$  is again given by (40). In conclusion, the condition (b) from the U.A.S. definition is satisfied and the proof is complete.

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#### STABILITATEA UNIFORM ASIMPTOTICĂ PENTRU SISTEMLILE DE ECUAȚII INTEGRALE VOLTERRA

(Rezumat)

Prin analogie cu teoria stabilității din cazul sistemelor diferențiale ordinare, se definește un nou fel de stabilitate pentru soluția unui sistem (liniar sau neliniar) de ecuații Volterra. Propoziția 4 precizează un caz în care stabilitatea uniform asimptotică se transmite de la sistemul liniar (14) la sistemul perturbat (31).



## UN PROCESO ESTOCASTICO TRIDIMENSIONAL DE NATALIDAD, MORTALIDAD E INMIGRACION

POR

DARIO MARAVALL CASESNOVES

Sea una población cuyo tamaño varía aleatoriamente en el tiempo, de acuerdo con el siguiente modelo probabilístico: si en el instante  $t$  el número de individuos que componen la población es  $n$ , en el instante  $t + dt$ , siendo  $dt$  un intervalo infinitesimal de tiempo, la población puede estar formada por  $n + 1$ ,  $n$  o  $n - 1$ , individuos, según las probabilidades respectivas

$$(1) \quad (\lambda n + \nu) dt; \quad 1 - (\lambda n + \nu + \mu n) dt; \quad \mu n dt$$

la primera de que nazca un individuo en el intervalo infinitesimal de tiempo de  $t$  a  $t + dt$ , la tercera de que muera un individuo, y la segunda de que no nazca ni muera ningún individuo, es decir de que la población permanezca estacionaria. Se trata de un proceso estocástico unidimensional de natalidad y mortalidad. Ahora bien, este proceso se puede transformar en un proceso estocástico bidimensional de natalidad, si consideramos en vez de la variable aleatoria (v.a.)  $n$ , población viva, las dos v.a.  $y, z$ , que representan los números de individuos nacidos y muertos hasta la fecha. Se tiene que

$$(2) \quad n = y - z$$

y mientras  $n$  es una v.a. que puede aumentar, disminuir o permanecer estacionaria, por el contrario  $y, z$  son v.a. que solamente pueden aumentar o permanecer estacionarias, pero en ningún caso disminuir. Se transforma de este modo un proceso estocástico unidimensional de natalidad y mortalidad en otro bidimensional de solo natalidad, porque la mortalidad biológica, es en este último caso una natalidad matemática, es decir una v.a. monótona no decreciente.

Si representamos por  $P(y, z, t)$  la probabilidad de que en el tiempo  $t$  hayan nacido  $y$  individuos y muerto  $z$  ( $y - z$  sería el número de vivos), por las reglas de las probabilidades totales y compuestas, se cumple que

$$(3) \quad \begin{aligned} P(y, z, t + dt) = & P(y - 1, z, t) (\nu + \lambda | y - z - 1) dt + \\ & + P(y, z - 1, t) \mu (y - z + 1) dt + \\ & + P(y, z, t) (1 - (\nu + (\lambda + \mu) (y - z))) dt, \quad 0 \leq z \leq y, \end{aligned}$$

de la que se sigue que

$$(4) \quad \begin{aligned} P'(y, z, t) = & P(y-1, z, t)(\nu + \lambda(y-z-1)) + \\ & + P(y, z-1, t)\mu(y-z+1) - P(y, z, t)(\nu + \\ & + (\lambda + \mu)(y-z)), \quad 0 \leq z \leq y, \end{aligned}$$

la desigualdad que figura en (3) y (4) es debido a que siempre el número de muertos es igual o inferior al de nacidos. Si suponemos que en el instante inicial  $t = 0$ , la población es nula, la ecuación diferencial inicial de la cadena (4) es la

$$(5) \quad P'(0, 0, t) = -\nu P(0, 0, t), \quad P(0, 0, 0) = 1.$$

Para resolver este sistema de ecuaciones diferenciales, introduzcamos la función generatriz (f.g.)

$$(6) \quad g(v, w, t) = \sum_{y=0}^{\infty} \sum_{z=0}^y v^y w^z P(y, z, t),$$

si multiplicamos ambos miembros de (3) y (4) por

$$(7) \quad v^y w^z$$

y efectuamos la sumación indicada en (6), se obtiene a partir de esta operación la ecuación en derivadas parciales

$$(8) \quad \frac{\partial g}{\partial t} = (\lambda v - \lambda - \mu + \mu w) \left( v \frac{\partial g}{\partial v} - w \frac{\partial g}{\partial w} \right) + \nu(v-1)g,$$

que como consecuencia de (5) tiene que cumplir la condición inicial

$$(9) \quad g(v, w, 0) = 1.$$

La integración de la (8) se hace a partir del sistema auxiliar de ecuaciones diferenciales

$$(10) \quad -dt = \frac{dv}{(\lambda v - \lambda - \mu + \mu w)v} = \frac{-dw}{(\lambda v - \lambda - \mu + \mu w)w} = \frac{-dg}{\nu(v-1)g}$$

y un cálculo largo y difícil, que omitimos, nos lleva a la siguiente solución

$$(11) \quad g(v, w, t) = e^{-\nu t} \left[ \frac{v_1 - v_2}{(v - v_2)e^{-\lambda t v_1} - (v - v_1)e^{-\lambda t v_2}} \right]^{\nu/\lambda},$$

siendo  $v_1$  y  $v_2$  las dos raíces de la ecuación de segundo grado

$$(12) \quad \lambda x^2 - (\lambda + \mu)x + \mu v w = 0,$$

o sea

$$(13) \quad \frac{\lambda + \mu}{2\lambda} \pm \frac{\sqrt{(\lambda + \mu)^2 - 4\mu\lambda vw}}{2\lambda} \begin{matrix} \nearrow v_1 \\ \searrow v_2 \end{matrix}.$$

Si en (11) se hace



$$(14) \quad vw = 1$$

se obtiene la f.g. de la población viva  $n$ , por  $s$  ésta la diferencia entre las dos poblaciones de nacidos y muertos.

Habíamos supuesto que la población inicial era nula ; pero si suponemos que tiene un cierto valor  $a$ , entonces las v.a. de los números de nacidos y de muertos, se compone de la suma de las dos v.a., cuya f.g. es la (11), y de  $a$  v.a. que son soluciones del proceso estocástico que resulta de hacer  $v = 0$  y suponer la población inicial formada por un solo individuo. Por tanto la f.g. en este caso es el resultado de multiplicar (11) por

$$(15) \quad \left[ \frac{v_1 - v_2 \frac{v - v_1}{v - v_2} e^{\lambda(v_1 - v_2)t}}{1 - \frac{v - v_1}{v - v_2} e^{\lambda(v_1 - v_2)t}} \right]^a$$

(vease bibliografía). Esto nos permite resolver también el problema en el caso en que la población inicial es a su vez una v.a.

Pero se puede obtener un análisis más profundo de este problema, distinguiendo dentro de la población viva  $y$  los que han nacido como consecuencia del factor  $\lambda(y-z)$  y del factor  $v$ . A lo que se le puede dar una interpretación biológica, los primeros serían los individuos nacidos en el seno de la población, y los últimos los que llegarían a ella como inmigrantes desde poblaciones exteriores. De esta forma el proceso estocástico bidimensional anterior se transforma en otro tridimensional en el que  $z$ , sigue siendo el número de muertos, una nueva v.a. que designamos por  $x$  ( $x \leq y$ ) el número de inmigrantes y por lo tanto la diferencia  $y-x$  el número de individuos nacidos en el seno de la población. Si designamos por  $P(x, y, z, t)$  la probabilidad de que en el tiempo  $t$  las tres antedichas v.a. tengan los valores  $x, y, z$ , por las reglas de las probabilidades totales y compuestas se cumple que

$$(16) \quad \begin{aligned} P(x, y, z, t + dt) = \\ = P(x-1, y-1, z, t) v dt + P(x, y-1, z, t) \lambda(y-z-1) dt + \\ + P(x, y, z-1, t) \mu(y-z+1) dt + P(x, y, z, t) (1 - (v + (\lambda + \mu)(y-z)) dt), \end{aligned}$$

con las condiciones

$$(17) \quad x \leq y, \quad z \leq y;$$

y si hacemos la hipótesis de que

$$(18) \quad P(0, 0, 0, 0) = 1$$

el método de las f.g. seguido en el caso bidimensional nos conduce a la ecuación en derivadas parciales

$$(19) \quad \frac{\partial g}{\partial t} = (\lambda v - \lambda - \mu + \mu v) \left( v \frac{\partial g}{\partial v} - w \frac{\partial g}{\partial w} \right) + v(\mu v - 1)g.$$

Ahora la f.g. depende de tres variables :  $g(u, v, w)$  que corresponden a las tres v.a.  $x, y, z$ ; la (19) se puede resolver por procedimiento similar al de la (8) y su solución es

$$(20) \quad g(u, v, w, t) = e^{-vt} \left[ \frac{v_1 - v_2}{(v - v_2) e^{-\lambda t v_1} - (v - v_1) e^{-\lambda t v_2}} \right]^{\frac{\mu v}{\lambda}},$$

con el significado de  $v_1$  y  $v_2$  dado en (13).

Se puede descomponer así la natalidad  $y$  en natalidad propiamente dicha  $y - x$  e inmigración  $x$ .

Si en el proceso anterior hacemos  $v = 0$ , suprimimos la inmigración, la natalidad matemática coincide en este caso con la biológica. Si hacemos  $\mu = 0$  suprimimos la mortalidad y resolvemos el problema de poder distinguir en un proceso estocástico de natalidad entre sus dos componentes : natalidad propiamente dicha e inmigración. Si hacemos  $\lambda = 0$  suprimimos la natalidad propiamente dicha y queda reducido a un proceso con inmigración y mortalidad, que tiene la notable propiedad de que cuando el tiempo tiende a infinito, la f.g. tiende a la de la distribución de Poisson, de modo estable.

Como casos curiosos se pueden citar que si  $v = 0$  y si  $\lambda = \mu$ , al tender el tiempo  $t$  a infinito,  $z$  e  $y$  convergen en probabilidad y cada una de ellas converge a una v.a. cuya ley de probabilidad no pertenece al dominio de atracción de la ley normal, si no a la de Levy. Me parece que debe de ser uno de los escasísimos fenómenos naturales que gozan de esta propiedad, un ejemplo conocido de antes, que goza de esta propiedad es el número de partidas que transcurren entre dos empates sucesivos en el juego de cara y cruz.

Si  $v = 0$ , y  $\lambda > \mu$  en este caso tanto  $y$ , como  $z$  tienen probabilidades no nulas de valer infinito y las complementarias a la unidad de ser v.a. finitas.

Las aplicaciones a la Biología, Física, Sociología etc., de estos modelos de procesos estocásticos son muy numerosas, y tanto para su conocimiento, como para un estudio más profundo y detallado de los mismos, remitimos a la bibliografía que se cita.

En todo este trabajo natalidad y mortalidad en sentido matemático se definen como variables monótonas no decreciente y no creciente respectivamente. Por tanto los números de nacidos y de muertos en sentido biológico en una población viva, donde los números de individuos nacen y mueren, son ambas variables en sentido matemático natalidad.

En mi opinión, como consecuencia de estas investigaciones, el número electrones positivos en el seno de la materia, sigue una ley de probabilidad de Poisson, estacionaria en el tiempo (vease bibliografía).

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## UN PROCES ALEATOR TRIDIMENSIONAL DE NATALITATE, MORTALITATE ȘI IMIGRAȚIE

(Rezumat)

Se dă o metodă de transformare a proceselor aleatoare unidimensionale de natalitate și de mortalitate în procese bidimensionale de natalitate, și de descompunere a proceselor aleatoare unidimensionale de natalitate în bidimensionale de natalitate, prin descompunerea primelor în două componente. Se aplică rezultatele la un proces aleator tridimensional de natalitate și mortalitate, la care se descompune natalitatea în natalitate propriu-zisă (biologică) și imigrație.

Metodele prezentate prezintă aplicații în fizică, biologie, sociologie etc.





## SUR LA RÉPARTITION ASYMPTOTIQUE D'UN PROCESSUS ALÉATOIRE À TEMPS ALÉATOIRE

PAR

ELENA NENCIU

1. En utilisant la terminologie de [2], par processus aléatoire à temps discrètement aléatoire on comprend un processus aléatoire de forme  $\{y_{v_n}, 1 \leq n < +\infty\}$ , où  $\{Y_n, 1 \leq n < +\infty\}$  est une suite de variables aléatoires définies sur un champ de probabilité  $\{\Omega, \mathcal{X}, P\}$  et  $\{v_n, 1 \leq n < +\infty\}$  est une suite de variables aléatoires avec des valeurs entières positives définies aussi sur  $\{\Omega, \mathcal{X}, P\}$ . Les recherches consacrées à l'étude de la répartition asymptotique d'un processus aléatoire à temps aléatoire ont été effectuées dans deux directions. Dans les travaux de la première direction on a trouvé la forme de la répartition asymptotique du processus aléatoire à temps aléatoire  $\{Y_{v_n}, 1 \leq n < +\infty\}$  dans certaines conditions imposées aux suites de variables aléatoires  $\{Y_n, 1 \leq n < +\infty\}$  et  $\{v_n, 1 \leq n < +\infty\}$ . Le résultat le plus général obtenu dans cette direction est contenu dans [1]. Dans la seconde direction, plus étroite que la première, on a trouvé des conditions dans lesquelles la répartition asymptotique du processus aléatoire  $\{Y_n, 1 \leq n < +\infty\}$  reste invariante quand on passe du temps habituel, qui s'écoule d'une manière monotone vers l'infini, au temps aléatoire. Parmi les travaux qui s'encadrent dans la seconde direction on remarque [2].

2. Soit  $\{X_j, 1 \leq j < +\infty\}$  une suite de variables aléatoires indépendantes définies sur un champ de probabilité  $\{\Omega, \mathcal{X}, P\}$  et  $Y_n = \sum_{j=1}^n X_j$ ,  $1 \leq n < +\infty$ . Le théorème de [1] donne la forme de la répartition asymptotique du processus aléatoire à temps aléatoire  $\{Y_{v_n}, 1 \leq n < +\infty\}$ , convenablement normé, dans le cas où pour tout  $n$ , la variable aléatoire  $v_n$  est indépendante des variables aléatoires  $X_j, 1 \leq j < +\infty$ . On démontre par la suite un théorème analogue au théorème de [1], en renonçant à l'hypothèse d'indépendance du temps aléatoire  $v_n$  des variables aléatoires indépendantes  $X_j, 1 \leq j < +\infty$ .

**Théorème 1.** Soit  $\{X_j, 1 \leq j < +\infty\}$  une suite de variables aléatoires indépendantes définies sur un champ de probabilité  $\{\Omega, \mathcal{X}, P\}$ ,  $Y_n =$

$= \sum_{j=1}^n X_j$ ,  $1 \leq n < +\infty$ , et  $\{v_n, 1 \leq n < +\infty\}$  une suite de variables aléatoires avec des valeurs entières positives définies aussi sur  $\{\Omega, \mathcal{X}, P\}$ . Si

$$(2.1) \quad \lim_{n \rightarrow +\infty} P\left(\frac{Y_n - an^\alpha}{bn^\beta} < x, \frac{v_n - cn^\gamma}{dn^\delta} < y\right) = F(x, y)$$

dans tous les points de continuité de la fonction propre de répartition  $F(x, y)$ , où  $\alpha > \beta$ ,  $b \neq 0$ ,  $\gamma > \delta > 0$ ,  $d \neq 0$ , alors

$$(2.2) \quad \lim_{n \rightarrow +\infty} P\left(\frac{Y_{v_n} - gn^\lambda}{hn^\mu} < x\right) = G(x).$$

Soit  $U$  et  $V$  des variables aléatoires définies sur le champ de probabilité  $\{\Omega, \mathcal{X}, P\}$  ayant la fonction de répartition commune  $F(x, y)$ . Les constantes  $g, h, \lambda, \mu$  et la fonction de répartition  $G(x)$  de la relation (2.2) sont déterminées ainsi :

I. Si  $a \neq 0, c \neq 0$ , et  $\gamma\beta > \gamma(\alpha - 1) + \delta$  ou  $a = 0$ , mais  $c \neq 0$ , alors  $g = ac^\alpha$ ,  $h = bc^\beta$ ,  $\lambda = \gamma\alpha$ ,  $\mu = \gamma\beta$ ,  $G(x) = F(x, +\infty)$ .

II. Si  $a \neq 0, c \neq 0$  et  $\gamma(\alpha - 1) + \delta > \gamma\beta$ , alors  $g = ac^\alpha$ ,  $h = \alpha ac^{\alpha-1} d$ ,  $\lambda = \gamma\alpha$ ,  $\mu = \gamma(\alpha - 1) + \delta$ ,  $G(x) = F(+\infty, x)$ .

III. Si  $a \neq 0, c \neq 0$  et  $\gamma(\alpha - 1) + \delta = \gamma\beta$ , alors  $g = ac^\alpha$ ,  $h = 1$ ,  $\lambda = \gamma\alpha$ ,  $\mu = \gamma\beta$ ,  $G(x) = P\{\alpha ac^{\alpha-1} dV + bc^\alpha U < x\}$ .

IV. Si  $a \neq 0, c = 0$ , alors  $g = 0$ ,  $h = ad^\alpha$ ,  $\mu = \alpha\delta$ ,  $G(x) = P\{V^\alpha < x\}$ .

V. Si  $a = 0, c = 0$ , alors  $g = 0$ ,  $h = bd^\beta$ ,  $\mu = \beta\delta$ ,  $G(x) = P\{UV^\beta < x\}$ .

**Démonstration.** En tenant compte des résultats de [6], si la suite des fonctions de répartition correspondant aux vecteurs aléatoires  $(\xi_n, \eta_n)$ ,  $1 \leq n < +\infty$ , converge faiblement vers la fonction de répartition du vecteur aléatoire  $(\xi_0, \eta_0)$ , alors on peut construire une suite de vecteurs aléatoires  $\{(\bar{\xi}_n, \bar{\eta}_n), 0 \leq n < +\infty\}$  de sorte que les fonctions de répartition correspondant aux vecteurs aléatoires  $(\xi_n, \eta_n)$  et  $(\bar{\xi}_n, \bar{\eta}_n)$  coïncident pour tout  $n$ ,  $0 \leq n < +\infty$ , et que la suite de vecteurs aléatoires  $\{(\bar{\xi}_n, \bar{\eta}_n), 1 \leq n < +\infty\}$  converge presque sûrement vers le vecteur aléatoire  $(\varepsilon_0, \eta_0)$ . En effet, il suffit de choisir le champ de probabilité  $\{\Omega, \mathcal{X}, P\}$  de la manière suivante :  $\Omega = [0, 1]$ ,  $\mathcal{X} = \mathcal{B}_{[0,1]}$  ( $\sigma$ -algèbre des ensembles Borel de  $[0, 1]$ ) et  $P$  mesure Lebesgue sur  $[0, 1]$ . Ensuite on raisonne comme dans la démonstration du théorème de [1]. Soit  $\{(U_n, V_n), 1 \leq n < +\infty\}$  une suite de vecteurs aléatoires qui ont respectivement les mêmes fonctions de répartition que les vecteurs aléatoires de la suite

$$\left\{ \left( \frac{Y_n - an^\alpha}{bn^\beta}, \frac{v_n - cn^\gamma}{dn^\delta} \right), 1 \leq n < +\infty \right\}.$$

Soit  $F(x, y)$  la fonction de répartition du vecteur aléatoire  $(U, V)$  vers lequel converge presque sûrement la suite de vecteurs aléatoires  $\{(U_n, V_n), 1 \leq n < +\infty\}$ . Pour  $1 \leq n < +\infty$ , le vecteur aléatoire  $(\bar{Y}_n, v_n)$ , où  $\bar{Y}_n = an^\alpha + bn^\beta U_n$ ,  $v_n = cn^\gamma + dn^\delta V_n$ , a la même fonction de ré-



partition que le vecteur aléatoire  $(Y_n, v_n)$ . C'est pour cela que,  $\bar{Y}_{v_n}$  a la même fonction de répartition que  $Y_{v_n}$ , pour  $1 \leq n < +\infty$ . Par conséquent, l'étude de la répartition asymptotique du processus aléatoire  $\{Y_{v_n}, 1 \leq n < +\infty\}$  se réduit à l'étude de la répartition asymptotique du processus aléatoire  $\{\bar{Y}_{v_n}, 1 \leq n < +\infty\}$ . Évidemment, pour  $n \rightarrow +\infty$ , on a

$$(2.3) \quad \bar{Y}_n = an^\alpha + bn^\beta U + O(n^\beta), \quad \bar{v}_n = cn^\gamma + dn^\delta V + O(n^\delta).$$

De (2.3) il résulte

$$(2.4) \quad \bar{Y}_{v_n} = a[cn^\gamma + dn^\delta V + O(n^\delta)]^\alpha + b[cn^\gamma + dn^\delta V + O(n^\delta)]^\beta U + O\{[cn^\gamma + dn^\delta V + O(n^\delta)]^\beta\}.$$

En développant (2.4) d'après les puissances de  $n$  et en considérant uniquement les termes de degré supérieur on obtient tous les cinq cas du théorème.

**3.** Concernant la répartition asymptotique du processus aléatoire des sommes normées par des variables aléatoires indépendantes et identiquement distribuées, quand on passe du temps habituel au temps aléatoire, dans [3] on démontre le

**Théorème 2.** Soit  $\{X_j, 1 \leq j < +\infty\}$  une suite de variables aléatoires indépendantes, identiquement distribuées,  $M(X_j) = 0$ ,  $D^2(X_j) = 1$ ,  $1 \leq j < +\infty$ ,  $Y_n = \sum_{j=1}^n X_j / \sqrt{n}$ ,  $1 \leq n < +\infty$ . Si  $\{v_n, 1 \leq n < +\infty\}$  est une

suite de variables aléatoires avec des valeurs entières positives de sorte que la suite de variables aléatoires  $\{v_n/n, 1 \leq n < +\infty\}$  converge en probabilité vers une variable aléatoire positive quelconque  $v$ , alors a lieu

$$(3.1) \quad \lim_{n \rightarrow +\infty} P(Y_{v_n} < x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp\left(-\frac{t^2}{2}\right) dt.$$

**Démonstration.** Conformément au théorème limite centrale de Lindeberg, des hypothèses du Théorème 2, il résulte

$$(3.2) \quad \lim_{n \rightarrow +\infty} P(Y_n < x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp\left(-\frac{t^2}{2}\right) dt.$$

En appliquant le cas V du Théorème 1 pour  $a=0$ ,  $b=1$ ,  $c=0$ ,  $\beta=0$ ,  $\delta=1$  on obtient (3.1).

**4.** Soit la suite de nombres réels  $\{B_n, 1 \leq n < +\infty\}$  de sorte que

$$B_n = n^\alpha L(n), \quad (\alpha > 0), \quad \text{et} \quad \lim_{n \rightarrow +\infty} \frac{L([cn])}{L(n)} = 1$$

pour tout  $c > 0$ . Concernant la répartition asymptotique du processus aléatoire des sommes normées par de variables aléatoire indépendantes

et non identiquement distribuées, quand on passe du temps habituel au temps aléatoire, dans [4], on démontre le

**Théorème 3.** Soit  $\{X_j, 1 \leq j < +\infty\}$  une suite de variables aléatoires indépendantes, non identiquement distribuées,

$$M(X_j) = 0, D^2(X_j) < +\infty, 1 \leq j < +\infty, Y_n = \sum_{j=1}^n X_j, 1 \leq n < +\infty,$$

et  $B_n^2 = D^2(Y_n)$  où la suite de nombres réels  $\{B_n, 1 \leq n < +\infty\}$  satisfait aux conditions précédentes. Soit  $\{v_n, 1 \leq n < +\infty\}$  une suite de variables aléatoires avec des valeurs entières positives de sorte que la suite de variables aléatoires  $\{v_n/n, 1 \leq n < +\infty\}$  converge en probabilité vers une variable aléatoire positive quelconque  $v$ . Si

$$(4.1) \quad \lim_{n \rightarrow \infty} P(Y_n < B_n x) = F(x),$$

où  $F(x)$  est une fonction de répartition continue, alors a lieu

$$(4.2) \quad \lim_{n \rightarrow +\infty} P(Y_{v_n} < B_n x) = \int_0^{+\infty} F\left(\frac{x}{y^\alpha}\right) dH(y),$$

où  $H(y)$  est la fonction de répartition de la variable aléatoire  $v$ .

**Démonstration.** Conformément au Corollaire 1 du Théorème 4 de [5], on a

$$(4.3) \quad \lim_{n \rightarrow +\infty} P\left(\frac{Y_n}{B_n} < x, \frac{v_n}{n} < y\right) = F(x) H(y).$$

En appliquant le cas V du Théorème 1 pour  $a = 0, c = 0, d = 1, \alpha = \beta, \delta = 1$  on obtient (4.2).

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#### ASUPRA REPARTIȚIEI ASIMPTOTICE A UNUI PROCES ALEATOR CU TIMP ALEATOR

(Rezumat)

Se demonstrează câteva teoreme asupra comportării asimptotice a unui proces aleator cu timp aleator renunțând la ipoteza independenței timpului aleator de variabilele aleatoare care constituie procesul aleator.



# ON RANDOM INTEGRAL EQUATION OF MIXED TYPE

BY

MARICA LEWIN

The random integral equation of mixed type

$$x(t, \omega) = f(t, \omega) + \int_0^t h(t, s, x(s, \omega), \omega) ds + \int_0^\infty g(t, s, x(s, \omega), \omega) ds$$

has been studied by many authors.

They consider the integrals of the equation as Bochner integrals and they look for a solution on the space  $C = C(R_1, L_2(\Omega, A, P))$ .

Concerning this subject, we mention the papers of M. Rao, Prakasa Rao [2], Daley [4].

This Note deals with the equation

$$\begin{aligned} x(t, \omega) = f(t, \omega) + \int_0^t q_1(t, s, x(s, \omega), \omega) dw_s + \\ (1) \quad + \int_{0-\infty}^{t+\infty} \int_{-\infty}^{\infty} q_2(t, s, x(s_1, \omega), \omega, v) m(ds + dv) + \int_0^\infty q_3(t, s, x(s, \omega), \omega) dw_s + \\ + \int_{0-\infty}^\infty \int_{-\infty}^\infty q_4(t, s, x(s, \omega), \omega, u) m(ds \times du). \end{aligned}$$

Let  $(\Omega, A, P)$  denote a complete atomic probability space  $w(s)$ ,  $s \in [0, \infty)$  a Wiener process and  $m(ds \times dv)$ ,  $(s, v) \in [0, \infty) \times (-\infty, \infty)$  a random Poisson measure.

Let us suppose that a family of  $\sigma$ -algebras  $F_t$ ,  $t \in [0, \infty)$  included in  $A$  as defined, so that, the following conditions are fulfilled:

a) for  $t_1 < t_2$ ,  $F_{t_1} \subset F_{t_2}$ ;

b) No matter what  $t \in [t_0, \infty)$  as, the random variables  $w(t)$ ,  $m(\Delta \times A)$ , where  $\Delta = [t_0, t]$ ,  $ACR$  independent of each other, are measurable with respect to  $F_t$ ;

c) For  $t_1 < t_2 < \dots < t_r$ , the variables  $w(t_2) - w(t_1), \dots, w(t_r) - w(t_{r-1}), m[t_1, t_2] \times A_j, \dots, m[t_{r-1}, t_r] \times A_j, j = 1, k$ , where  $A_1, A_2, \dots, A_k$  are arbitrary Borel sets in  $R$  for which the corresponding variables are meaningful do not depend mutually on each of the events of the  $\sigma$ -algebra  $F_t$ .

A process  $x(t, \omega)$  defined on  $R^+ \times \Omega \rightarrow R$  is nonanticipating with respect to the family of  $\sigma$ -algebra  $F_t$ , if for every  $t \in [0, \infty)$ ,  $x(t, \omega)$  is  $F_t$  measurable.

Let  $C = C(R^+, L_2(\Omega, A, P))$  the space of all continuous and bounded functions family defined on  $R^+$ , with values in  $L_2(\Omega, A, P)$  nonanticipating with respect to the family of  $\sigma$ -algebra  $F_t, t \in [0, \infty)$ , that is the space of all continuous functions in mean square.

The norm of the space is defined by

$$\|x(t, w)\|_C = \sup_{t \geq 0} \|x(t, \omega)\|_2 = \sup_{t \geq 0} \int_{\Omega} |x(t, \omega)|^2 dP(\omega)^{1/2}.$$

Let  $C_c(R^+, L_2(\Omega, A, P))$  be the space of all continuous functions on  $R^+$ , with values in  $C_2(\Omega, A, P)$ . The topology of the space is defined by the family of seminorms

$$x(t, \omega)_n = \sup_{0 \leq t \leq n} \int_{\Omega} |x(t, \omega)|^2 dP(\omega)^{1/2} = \sup_{0 \leq t \leq n} \{E |x(t, \omega)|^2\}^{1/2}, \quad n = 1, 2, \dots$$

The fact that the process  $x(t, \omega)$  belongs to  $C_c(R^+, L_2)$  implies that

$\int_0^T |x(t, \omega)|^2 dt < \infty, \forall T \in [0, \infty)$  with probability 1. The integral is considered in the Lebesgue sense.

$x(t, \omega) \in C_c(R^+, L_2)$  implies  $x^2(t, w)$  is continuous in probability. A theorem of Skorokhod states that for a process, continuous in probability in all points  $t \in [0, \infty)$  with the exception of a finite number of points, there exists a measurable, separable process, equivalent to the process  $x(t, \omega)$ . If  $x^2(t, \omega)$  is measurable in  $t$ , for  $t \in [0, T], \forall T \geq 0$ , due to Fubini's

theorem  $\int_0^t |x^2(t, \omega)|^2 dt < \infty, \forall t \in R^+$  in the Lebesgue sense is valid

with probability 1.

We suppose that  $f(s, \omega), q_1(t, s, x, w), q_2(t, s, x, u, \omega), q_3, q_4$  are measurable, for every  $s \in [0, \infty), x \in R, u \in R$ , with respect to  $F_s$ .

Since  $F_s \leq F_t$ , for  $s \leq t$ , they are also  $F_t$  measurable, for every  $t > s$ . The functions  $q_3, q_4$  must be  $F_t$  measurable also for  $t < s$ .

**Definition 1.** A process  $x(t, \omega)$  is a solution of the equation (1), if for every  $t \in R^+, x(t, \omega), L_2(\Omega, A, P)$  and  $x(t, \omega)$  satisfies (1) with probability 1.

For proving the existence and uniqueness of the equation (1) we shall use the fixed point theorem due to Krasnoselski.



Let  $E$  be a closed, bounded and convex subset of a separable Banach space  $X$ . Let  $T, S$ , be continuous operators from  $X$  into  $X$ , so that

$$Sx + Ty \in E \text{ for all } x, y \in E;$$

there exists a real constant  $k$  such that

$$\|Sx - Sy\| \leq k \|x - y\|, \text{ for all } x, y \in E, \quad 0 < k < 1;$$

$T$  is a compact operator.

Then, there exists an element from  $E$ , which verifies the equation

$$S\mathcal{L} + T\mathcal{L} = \mathcal{L}.$$

We assume the following conditions:

A)  $f \in C(R^+, L_2)$

(I) for  $0 \leq s \leq t < \infty$ , and  $x(s, \omega) \in L_2$ ,

$$q(t, s, x(s, \omega), \omega) \in L_2(\Omega, A, P), \quad i = 1, 2, \dots;$$

(II) for  $0 < s, t < \infty$ ,  $x(s) \in L_2$ ,  $q_1(t, s, x(s, \omega)) \in L_2$ ,  $i = 3, 4, \dots$ ;

(III) for  $s > t$ ,  $q_1(t, s, x, \omega) = q_2(t, s, x(s, \omega), \omega, v) = 0$ ;

(IV)  $q(t, s, 0, \omega) = 0$ , for  $x = 0$ ,  $q_1(t, s, 0, \omega, v) = 0$ ,  $i = 3, 4$ ,

with probability 1.

B) There exists nonnegative measurable functions  $k_V(t, s)$ ,  $k_F(t, s)$  so that

(I) for every  $\varepsilon > 0$ , there exists a  $\delta = \delta(\varepsilon)$ , such that for all  $x, y \in C$ ,

with  $\|x(t, \omega)\|_C \leq \delta$ ,  $\|y(t, \omega)\|_C < \delta$ ,

$$\|q_1(t, s, x(s, \omega), \omega) - q_1(t, s, y(s, \omega), \omega)\|_2^2 +$$

$$+ \int_{-\infty}^{+\infty} \|q_2(t, s, x(s, \omega), v) - q_2(t, s, y(s, \omega), v)\|_2^2 \frac{dv}{v^2} \leq$$

$$\leq \varepsilon k_V(t, s) \|x(t, \omega) - y(t, \omega)\|_2^2;$$

(II) for every  $\varepsilon > 0$ , there exists  $\delta = \delta(\varepsilon) > 0$ , such that for all  $x, y$  in  $C$ , with  $\|x(t, \omega)\| < \varepsilon$ ,

$$\|q_3(t, s, x(s, \omega), \omega) - q_3(t, s, y, \omega)\|_2^2 +$$

$$+ \int_{-\infty}^{+\infty} \|q_4(t, s, x, \omega, v) - q_4(t, s, y, \omega, v)\|_2^2 \frac{dv}{v^2} \leq \varepsilon k_F(t, s) \|x(t) - y(t)\|_2^2;$$

$$(III) \quad \sup_{0 < t < \infty} \int_0^t k_V(t, s) ds \leq k_V < \infty;$$

$$(IV) \quad \sup_{0 < t < \infty} \int_0^\infty k_F(t, s) ds \leq k_F < \infty;$$

(V) For every  $\varepsilon > 0$ , and  $t \in R^+$ ,

$$\int_0^{\infty} \left\{ \|q_3(t + \tau, s, x, \omega) - q_3(t, s, x, \omega)\|_2^2 + \right. \\ \left. + \int_{-\infty}^{\infty} \|q_4(t + \tau, s, x, v) - q_4(t, s, x, v, \omega)\|_2^2 \frac{dv}{v^2} \right\} ds.$$

When  $|\tau| \rightarrow 0$ , uniformly for all  $x \in C$ ,  $\|x(t, \omega)\|_C < \infty$ .

**Theorem 2.** Under these assumptions there exists  $\varepsilon_0 > 0$ , such that if  $\varepsilon \in (0, \varepsilon_0)$  there exists  $\delta > 0$  implying a unique solution  $x(t, \omega)$  of (1), for which  $\|x(t, \omega)\|_C \leq \varepsilon$ , provided that  $\|f(t, \omega)\|_C < \delta$ .

*Remark.* The stated conditions ensure the existence of the integrals for all  $t$  belonging to  $[0, \infty)$ .

*Proof.* Let  $\varepsilon_1$  be given, with  $\varepsilon_1 k_F < 1$ . Owing to the assumptions there is a  $\delta_1 > 0$  so that for  $\|x(t, \omega)\|_C \leq \delta$

$$\|q_3(t, s, \dots)\|_2^2 + \int_{-\infty}^{+\infty} \|q_4(t, s, x, \omega, v)\|_2^2 \leq \varepsilon k_F(t, s) \|t, \omega\|_2^2.$$

Let  $\varepsilon_2 = (1 - \varepsilon_1 k_F)/2k_V$  and choose  $\delta_2 > 0$  such that (BI) holds for all  $\|x(t, \omega)\|_C \leq \delta_2$ ,  $\|y(t, \omega)\|_C \leq \delta_2$ . Let  $\varepsilon_0 = \min\{\delta_1, \delta_2\}$ .

Let  $T_V, T_F$  the operators from  $C \rightarrow C$  defined by

$$(T_V x)(t, \omega) = f(t, \omega) + \int_0^{\infty} q_1(t, s, x, \omega) dw_s + \int_0^{t+\infty} \int_{-\infty}^{+\infty} q_2(t, s, x, \omega, v) m(ds \times dv), \\ (T_F x)(t, \omega) = \int_0^{\infty} q_3(t, s, x, \omega) dw_s + \int_0^{t+\infty} \int_{-\infty}^{+\infty} q_4(t, s, x, \omega, v) m(ds \times dv).$$

The aim of proving is to show that Krasnoselski's conditions are fulfilled.

We shall prove for  $T_V$ , that is  $\forall \varepsilon > 0$ ,  $\exists \delta(\varepsilon) > 0$  such that  $\|x - y\|_C < \delta$ ,  $\|T_V x - T_V y\|_C < \varepsilon$ .

$$(2) \quad \|T_V x - T_V y\|_2^2 = \int_0^t E[q_1(t, s, x, \omega) - q_1(t, s, y, \omega)]^2 ds + \\ + \int_0^t \int_{-\infty}^{+\infty} E[q_2(t, s, x, \omega, v)^2 - q_2(t, s, y, \omega, v)^2] \frac{dv ds}{v^2}.$$

This equality is due to the properties of Wiener and Poisson integrals and to the independence of  $q_1$  and  $q_2$ , assumed in the hypotheses; (2) becomes, due to (II),



$$\|T_V x - T_V y\|_2^2 \leq \int_0^t \varepsilon k_V(t, s) \|x - y\|_2^2 \leq \varepsilon k_V,$$

which implies  $\|T_V x - T_V y\|_C \leq \sqrt{\varepsilon k_V} \|x - y\|_C$ .

As soon as  $\delta < \frac{\varepsilon}{\sqrt{\varepsilon k_V}} \|T_V x - T_V y\| \leq \varepsilon$ .

The continuity of  $T_F$  is shown in the same manner.

Now, let  $\rho > 0$  be and  $S(\rho) = \{x(t, \omega), C, \|x(t, \omega)\|_C \leq \rho\}$ ;  $S(\rho)$  is a bounded, convex closed set in  $C$ .

We consider  $S(\varepsilon) = \{x(t, \omega) \in C, \|x(t, \omega)\|_C < \varepsilon\}$  and show that there exists a  $\delta(\varepsilon)$  so that  $T_V x + T_F y \in S(\varepsilon)$  as soon as  $\|f(t, \omega)\|_C \leq \delta$

$$\begin{aligned} \|T_V x + T_F y\|_2^2 &\leq \|f(t, \omega)\|_2^2 + \int_0^1 \|q_1(t, s, x, \omega)\|_2^2 + \\ &+ \int_{-\infty}^{+\infty} \|q_2(t, s, x, \omega, v)\|_2^2 \frac{dv}{v^2} ds + \int_0^t \|q_3(t, s, x, \omega)\|_2^2 + \\ &+ \int_{-\infty}^{+\infty} \|q_4(t, s, x, \omega, v)\|_2^2 \frac{dv}{v^2} ds \leq (\varepsilon_2 k_V + \varepsilon_1 k_F) \|x(t, \omega)\|_C^2 = \\ &= \delta^2 + \left( \frac{1 - \varepsilon_1 k_F}{2} + \varepsilon_1 k_F \right)^2 < \varepsilon^2, \text{ as soon as } \delta < \varepsilon \sqrt{1 - \varepsilon_1 k_F}. \end{aligned}$$

It is easy to show that  $\forall x, y \in S(\varepsilon)$ ,  $T_V$ ;  $T_V$  is a contraction and  $T_F$  is a compact operator on  $S(\varepsilon)$  because  $T_F$  is equi continuous and uniformly bounded.

$$\begin{aligned} \|(T_F x)(t + \tau) - (T_F x)(t)\|_2^2 &\leq \int_0^\infty E[q_3(t + \tau, x, \omega) - q_3(t, x, \omega)]^2 + \\ &+ \int_{-\infty}^\infty \frac{E[q_4(t + \tau, x, \omega, v) - q_4(t, x, \omega, v)]^2 dv}{v^2} ds, \end{aligned}$$

which converges to 0, when  $|\tau| \rightarrow 0$ , uniformly with respect to  $x$ , for every  $t \in \mathbb{R}^+$ , which implies the equicontinuity. Also,

$$\|T_F x\|_2^2 = \int_0^\infty \left( E q_3^2 + \int_{-\infty}^\infty \frac{E q_v^2}{v^2} dv \right) ds \leq \varepsilon_1 k_F \varepsilon^2,$$

which implies  $\|T_F x\|_C < \varepsilon \sqrt{\varepsilon_1 k_F}$ .

Applying Krasnoselski's theorem, there is at least a solution  $x \in S(\varepsilon)$  of the equation  $T_V x(t, \omega) + T_F x(t, \omega) = x(t, \omega)$ , that is  $x(t, \omega)$  verifies the equation (1) almost surely.

Finally, to show the uniqueness of the solution in  $S(\varepsilon)$  suppose that there exists  $\omega$  random solution  $x_1(t, \omega)$ ,  $x_2(t, \omega)$  and let  $z(t, \omega) = x_1 - x_2$ .

Applying the hypotheses, we obtain  $\|z(t, \omega)\|_2^2 \leq \|z(t, \omega)\|_C^2 / (1 + \varepsilon_1 k_F)_2$  and so  $\|z(t, \omega)\|_C \leq \|z(t, \omega)\|_C \sqrt{(1 + \varepsilon_1 k_F)/2}$ . Since  $\varepsilon_1 k_F < 1$ , we obtain a contradiction.

*Remark.* a) One specializes the abstract operators of the equations to obtain the random integral equation of Volterra Fredholm type

$$(3) \quad \begin{aligned} x(t, \omega) = & f(t, \omega) + \int_0^t k_1(t, s, \omega) q_1(s, x) dw_s + \\ & + \int_0^t \int_{-\infty}^{+\infty} k_1(t, s, u) q_2(s, x, v) m(ds \times dv) + \\ & + \int_0^\infty k_2(t, s, \omega) h_1(s, x) d\omega_s + \int_0^t \int_{-\infty}^{+\infty} k_2(t, s, \omega) h_2(s, x, v) m(ds \times dv). \end{aligned}$$

b) It is interesting to consider stochastic equation with random parameter that is the equation

$$(4) \quad \begin{aligned} x(t, u, u') = & f(t, \omega) + \int_0^t q_1(t, s, x, \omega) dw_s + \\ & + \int_0^s \int_{-\infty}^{+\infty} q_2(t, s, x, u', v) m(du \times ds) + \\ & + \int_0^\infty q_3(t, s, x, \omega') d\omega_s + \int_0^\infty \int_{-\infty}^{+\infty} q_4(t, s, x, \omega', v) m(dv \times ds), \end{aligned}$$

where  $(\Omega, A, P)$  has the known meaning, and  $\omega' \in \Omega$ . For every  $\omega' \in \Omega$ ,  $x(t, \omega, \omega')$  is a random variable. Recalling the known assumptions we can define the random operators

$$T_V(\omega') = f(t, \omega) + \int_0^t q_1(t, s, x, \omega') dw_s + \int_0^t \int_{-\infty}^{+\infty} q_2(t, s, x, v, \omega') m(dv \times ds)$$

and

$$T_F(\omega') = \int_0^\infty q_3(t, s, x, \omega') d\omega_s + \int_0^\infty \int_{-\infty}^{+\infty} q_4(t, s, x, v, \omega') m(dv \times ds),$$

where

$$T_V \Omega \times C \rightarrow C, \quad T_F \Omega \times C \rightarrow C.$$



**Definition.** Let  $E, F$  be two normed linear spaces. A random operator  $T: \Omega \times E \rightarrow F$  is compact if for any bounded sequence  $\{x_n\}$  in  $E$  there is a subsequence  $\{x_{n_k}\}$  such that the sequence  $\{T(\omega') x_{n_k}\}$  is almost surely convergent in  $F$ .

For proving the existence of a solution for the equation (4) we use the following random analogue of Krasnoselski's fixed point theorem due to Prakasa Rao.

**Theorem 3.** Let  $(\Omega, A, P)$  be an atomic probability space, and let  $E$  be a closed, bounded, and convex subset of a separable Banach space  $X$ . Let  $T, S$  be continuous random operators from  $\Omega \times X$  into  $X$  such that

(I)  $[S(\omega')x + T(\omega')y] \in E$  for all  $x, y \in E$ , and  $\omega' \in \Omega$ ;

(II) there exists a real-valued random variable  $K(\omega)$ , such that  $\|S(\omega')x - S(\omega')y\| \leq K(\omega')\|x - y\|$  for all  $x, y \in E$ , where  $0 < K(\omega') < 1$  for all  $\omega' \in \Omega$  and

(III)  $T(\omega')$  is a compact operator for each  $\omega' \in \Omega$ .

Then, there exists a function defined on  $\Omega$ , with values in  $E$ , such that  $S(\omega')\mathcal{L}(\omega') + T(\omega')\mathcal{L}(\omega') = \mathcal{L}(\omega')$  for all  $\omega' \in \Omega$ .

If we maintain the assumptions made, only the functions  $k_V(t, s)$ ,  $k_F(t, s)$  will become  $k_V(t, s, \omega')$ ,  $k_F(t, s, \omega')$  and  $\int_0^\infty k_V(t, s, \omega') ds < k_F$  almost every

there, and all the conditions are satisfied for  $\omega' \in \Omega_0$  with  $P(\Omega_0) = 1$ , one shows easily (in the same manner we studied the equation (1)) that the operators  $T(\omega')$  and  $T_F(\omega')$  are continuous from  $C$  into  $C$ , with probability 1,  $T_V(\omega')$  is a random contraction in some conditions and  $T_F$  is a random compact operator.

Thus, the equation (1) has solution  $\mathcal{L}(t, \omega, \omega')$  for every  $\omega'$ , belonging to  $C$ .

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## ASUPRA UNEI ECUAȚII INTEGRALE STOCHASTICE DE TIP MIXT

(Rezumat)

Se demonstrează existența și unicitatea soluției ecuației integrale stochastice de tip mixt (1), utilizând în acest scop teorema de punct fix a lui Krasnoselski.

**Definition.** Let  $E, F$  be two normed linear spaces. A random operator  $T: \Omega \times E \rightarrow F$  is called *linear* if for any bounded sequence  $\{x_n\}$  in  $E$  there is a subsequence  $\{x_{n_k}\}$  such that the sequence  $\{T(x_{n_k})\}$  is almost surely convergent in  $F$ .

For proving the existence of a solution for the equation (1) we use the following random analogue of Krasnoselski's fixed point theorem due to Prokhorov [1].

**Theorem 2.** Let  $\Omega, E, F$  be an abstract probability space and let  $E$  be a closed, bounded, and convex subset of a separable normed space. Let  $T, S$  be continuous random operators from  $\Omega \times E$  into  $E$  such that

- (I)  $\|T(\omega)x + S(\omega)y - T(\omega)z - S(\omega)v\| \leq K(\omega)\|x - z\| + L(\omega)\|y - v\|$  for all  $x, y, z, v \in E$ , where  $0 < K(\omega) < 1$  for all  $\omega \in \Omega$  and
- (II) there exists a random variable  $\delta(\omega)$  such that  $\delta(\omega) > 0$  for all  $\omega \in \Omega$  and

(III)  $T(\omega)$  is a compact operator for each  $\omega \in \Omega$ .  
Then there exists a function defined on  $\Omega$  with values in  $E$  such that  $S(\omega)z(\omega) + T(\omega)z(\omega) = z(\omega)$  for all  $\omega \in \Omega$ .

If we maintain the designations made, only the functions  $K(\omega), L(\omega)$  will become  $K(\omega, x, y), L(\omega, x, y)$  and  $\int_0^1 K(\omega, x, y) dx < 1$  almost everywhere.

Here, and all the conditions are satisfied for  $\omega \in \Omega$  with  $P(\Omega) = 1$ , one shows easily (in the same manner we studied the equation (1)) that the operators  $T(\omega)$  and  $S(\omega)$  are continuous from  $E$  into  $E$  with probability 1.  $T, S$  is a random contraction in some conditions and  $T$  is a random compact operator.

Thus the equation (1) has solution  $z(\omega, x)$  for every  $\omega$ , belonging to  $E$ .

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## ASSEMBLY LINE RECTANGULAR INTEGRAL PROBABILITIES OF THE MIX

(Received)

So far, the existence of a solution for the equation (1) has been proved. The next step is to prove the existence of a solution for the equation (1) in the case of a random operator.



## ON A CLASS OF FINSLER CONNECTIONS

BY

RADU MIRON

1. To a Finsler space there are associated by means of geometrical criteries a number of important connections, called Cartan, Berwald, Rund, Hashiguchi, Matsumoto etc. connections. Between them the metrical connections have a special interest [2].

A. Kawaguchi, in the paper [1] determines the horizontal part of a metrical connection, canonical associated to any Finsler connection. He shows that the metrical connection associated with the Berwald connection is just the Cartan connection.

The author, in a joint paper with M. Hashiguchi [4], determined the vertical part of the metrical connection associated to a Finsler connection and proved that the metrical connection associated to the Rund or Hashiguchi connection coincides with the Cartan connection.

In this Note, we determine the set of Finsler connections which have the property that the associated metrical connection coincides with Cartan connection and establish some of their properties.

2. Let  $F(x, y)$  be the fundamental function and  $g_{ij}(x, y)$  the fundamental metric tensor field of a Finsler space  $F_n$ . A Finsler connection  $F\Gamma = (N, F, C)$  is called a metrical connection if:

$$(1) \quad g_{ij|k} = 0, \quad g_{ij|k} = 0.$$

Cartan connection,  $C\overset{\circ}{\Gamma} = (\overset{\circ}{N}, \overset{\circ}{F}, \overset{\circ}{C})$  is a metrical one. In the paper [4] we prove

**Proposition 1.** For every Finsler connection  $F\overset{\circ}{\Gamma} = (\overset{\circ}{N}, \overset{\circ}{F}, \overset{\circ}{C})$  in  $F_n$ , the connection  $K\overset{\circ}{\Gamma} = (\overset{\circ}{N}, \overset{\circ}{F}, \overset{\circ}{C})$  given by

$$(2) \quad \overset{\circ}{N}_j^i = \overset{\circ}{N}_j^i, \quad \overset{\circ}{F}_{jh}^i = \overset{\circ}{F}_{jh}^i + \frac{1}{2} g^{im} g_{jm|h}, \quad \overset{\circ}{C}_{jh}^i = \overset{\circ}{C}_{jh}^i + \frac{1}{2} g^{im} g_{jm|h}$$

is a metrical connection.

$K\overset{\circ}{\Gamma}$  is called Kawaguchi connection associated to  $F\overset{\circ}{\Gamma}$ . The horizontal part of this connection has been determined by prof. A. Kawaguchi in the paper [1].

Let

$$(3) \quad \Omega_{hk}^{ij} = \frac{1}{2}(\delta_h^i \delta_k^j - g_{hk} g^{ij}), \quad \Omega_{hk}^{*ij} = \frac{1}{2}(\delta_h^i \delta_k^j + g_{hk} g^{ij}),$$

be Obata operators [4], and

$$(4) \quad \mathcal{X} = \{F\overset{\circ}{\Gamma} \mid K\overset{\circ}{\Gamma} = C\overset{\circ}{\Gamma}\}.$$

**Theorem 2.** All connections of  $\mathcal{X}$  are given by

$$(5) \quad \overset{\circ}{N}_j^i = \overset{c}{N}_j^i, \quad \overset{\circ}{F}_{jk}^i = \overset{c}{F}_{jk}^i + \Omega_{hj}^{*im} X_{mk}^h, \quad \overset{\circ}{C}_{jk}^i = \overset{c}{C}_{jk}^i + \Omega_{hj}^{*im} Y_{mk}^h,$$

where  $X_{jk}^i, Y_{jk}^i$  are the arbitrary Finsler tensor fields.

**Proof.** Let  $(A, B, D)$  be the difference tensors of the pair  $(F\overset{\circ}{\Gamma}, C\overset{\circ}{\Gamma})$ :

$$(6) \quad \overset{\circ}{N}_j^i = \overset{c}{N}_j^i - A_j^i, \quad \overset{\circ}{F}_{jk}^i = \overset{c}{F}_{jk}^i + \overset{c}{C}_{jm}^i A_k^m - B_{jk}^i, \quad \overset{\circ}{C}_{jk}^i = \overset{c}{C}_{jk}^i - D_{jk}^i.$$

From (2), we have

$$\overset{k}{N}_j^i = \overset{c}{N}_j^i - A_j^i, \quad \overset{k}{F}_{jk}^i = \overset{c}{F}_{jk}^i + \overset{c}{C}_{jm}^i A_k^m - \Omega_{hj}^{im} B_{mk}^h, \quad \overset{k}{C}_{jk}^i = \overset{c}{C}_{jk}^i - \Omega_{hj}^{im} D_{mk}^h.$$

The equality  $K\overset{\circ}{\Gamma} = C\overset{\circ}{\Gamma}$  holds iff

$$A_j^i = 0, \quad \Omega_{hj}^{im} B_{mk}^h = 0, \quad \Omega_{hj}^{im} D_{mk}^h = 0.$$

It follows that  $F\overset{\circ}{\Gamma} \in \mathcal{X}$ , if and only if,  $F\overset{\circ}{\Gamma} = F\overset{\circ}{\Gamma}(\overset{c}{N})$  and  $B, D \in \text{Ker } \Omega$ . Therefore the difference tensors are given by

$$A_j^i = 0, \quad B_{jk}^i = -\Omega_{hj}^{*im} X_{mk}^h, \quad D_{jk}^i = -\Omega_{hj}^{*im} Y_{mk}^h,$$

$X_{jk}^i, Y_{jk}^i$  being the arbitrary tensor fields. If we substitute these values of  $(A, B, D)$  in (6), we have the equalities (5). Q.E.D.

**Proposition 3.** The set  $\mathcal{X}$  contains the connections given by the following values of the fields  $X_{jk}^i, Y_{jk}^i$ :

| $X_{jk}^i$              | $Y_{jk}^i$               | $F\overset{\circ}{\Gamma}$                         |
|-------------------------|--------------------------|----------------------------------------------------|
| 0                       | 0                        | $C\overset{\circ}{\Gamma}$ — Cartan connection     |
| $\overset{c}{P}_{jk}^i$ | $-\overset{c}{C}_{jk}^i$ | $B\overset{\circ}{\Gamma}$ — Berwald connection    |
| 0                       | $-\overset{c}{C}_{jk}^i$ | $R\overset{\circ}{\Gamma}$ — Rund connection       |
| $\overset{c}{P}_{jk}^i$ | 0                        | $H\overset{\circ}{\Gamma}$ — Hashiguchi connection |

**Proposition 4.** The set  $\mathcal{X}$  contains a single metrical connection. This is Cartan connection,  $C\overset{\circ}{\Gamma}$ .



Indeed, (1) and (5) have as a consequence

$$\Omega_{hj}^{*im} X_{mk}^h = 0, \quad \Omega_{hj}^{*im} Y_{mk}^h = 0.$$

Thus  $F\dot{\Gamma} = C\Gamma$ .

**Proposition 5.** *All the connections of  $\mathcal{X}$  have the property*

$$S_{(i,j,k)} g_{ih} T_{jk}^h = 0, \quad S_{(i,j,k)} g_{ih} \dot{S}_{jk}^h = 0.$$

This follows by a direct calculus from (5).

Let  $\mathcal{X}^* \subset K$  be the subset obtained for the values

$$X_{jk}^i = \delta_j^i \sigma_k + 2\delta_k^i \tau_j, \quad Y_{jk}^i = \delta_j^i \lambda_k + 2\delta_k^i \mu_j.$$

All the connections of  $\mathcal{X}^*$  are given by

$$(7) \quad \dot{N}_j^i = \dot{N}_j^i, \quad \dot{F}_{jk}^i = \dot{F}_{jk}^i + \delta_j^i \sigma_k + \delta_k^i \tau_j + g_{jk} \tau^i, \quad \dot{C}_{jk}^i = \dot{C}_{jk}^i + \delta_j^i \lambda_k + \delta_k^i \mu_j + y_{jk} \mu^i.$$

**Proposition 6.** *The connections of  $\mathcal{X}^*$  have the property*

$$g_{ij}|_k = -2(g_{ij} \sigma_k + g_{jk} \tau_i + g_{ki} \tau_j), \quad g_{ij}|_k = -2(g_{ij} \lambda_k + g_{jk} \mu_i + g_{ki} \mu_j).$$

**Proposition 7.** *The connections of  $\mathcal{X}^*$  are semi-symmetric.*

**3. Remarks.** 1. The following subsets of connections of  $\mathcal{X}^*$  are interesting: a)  $\tau_k = \mu_k = 0$ ; b)  $\sigma_k = \lambda_k = 0$ ; c)  $\sigma_k = \tau_k, \lambda_k = \mu_k$ .

2. The formulas (5) determine a group of transformations on  $\mathcal{X}$ . In the same way (7) determine a group of transformation on  $\mathcal{X}^*$ . It is necessary to study the invariants of these groups [3].

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## ASUPRA UNEI CLASE DE CONEXIUNI FINSLER

(Rezumat)

Se consideră un spațiu Finsler  $F_n$  și se determină clasa  $\mathcal{X}$  a conexiunilor Finsler  $F\dot{\Gamma}$  care au proprietatea că  $\chi\dot{\Gamma}$ , conexiunea metrică a lui Kawaguchi, coincide cu conexiunea Cartan  $C\Gamma$ . Se distinge de asemenea o subclasă  $\mathcal{X}^*$  remarcabilă din  $\mathcal{X}$ .





# KAWAGUCHI CONNECTIONS ASSOCIATED TO A FINSLER ALMOST SIMPLECTICAL STRUCTURE<sup>1)</sup>

BY

I. GHINEA

0. Having in mind the results of prof. R. Miron [5] we are dealing in this paper with the class of Finsler connections which have the property that Kawaguchi almost simplectical connection  $K\overset{\circ}{\Gamma}$  is the same with the canonical almost simplectical connection of  $a_{ij}(x, y)$ .

1. Let  $M$  be a  $2n$ -dimensional differential Finsler manifold. A Finsler almost simplectical structure on  $M$  is defined through a tensor field  $a_{ij}(x, y)$  ( $y$  being the support element in point  $x$ ) such that

- 1)  $a_{ij}(x, y) = -a_{ji}(x, y)$ ;
- 2)  $\det(a_{ij}(x, y)) \neq 0, \forall y \neq 0$ .

Let associate to the almost simplectical structure  $a_{ij}$  the following operators:

$$(1.1) \quad \theta_{kh}^{ij} = \frac{1}{2}(\delta_k^i \delta_h^j - a_{kh} a^{ji}), \quad \theta_{kh}^{*ij} = \frac{1}{2}(\delta_k^i \delta_h^j + a_{kh} a^{ji}).$$

A Finsler connection  $F\Gamma = (N, F, C)$  is called compatible with the almost simplectical structure  $a_{ij}(x, y)$  if

$$(1.2) \quad \begin{cases} \nabla_k a_{ij} = 0, \\ \dot{\nabla}_k a_{ij} = 0, \end{cases}$$

where  $\nabla_k, \dot{\nabla}_k$  are the covariant  $h$ -derivative and covariant  $v$ -derivative with respect to this connection. We then say that  $F\Gamma = (N, F, C)$  is an almost simplectical Finsler connection.

In the paper [1] we have proved the

**Theorem 1.1.** *All the Finsler connections  $(N_j^h, F_{jk}^h, C_{jk}^h)$  almost simplectical, are given by*

<sup>1)</sup> Presented at the National Conference on Geometry and Topology, Brashov, 2—4 June, 1979.

$$(1.3) \quad \begin{cases} N_j^h = \dot{N}_j^h + X_j^h, \\ F_{jk}^h = \dot{F}_{jk}^h + \frac{1}{2} a^{ph} \left( a_{pj} - X_k^q \frac{\partial a_{pj}}{\partial y^q} \right) + \theta_{jq}^{ph} U_{pk}^q, \\ C_{jk}^h = \dot{C}_{jk}^h + \frac{1}{2} a^{ph} \dot{\nabla}_k a_{pj} + \theta_{jq}^{ph} V_{pk}^q, \end{cases}$$

where

a)  $F\dot{\Gamma} = (\dot{N}_j^h, \dot{F}_{jk}^h, \dot{C}_{jk}^h)$  is a fixed Finsler connection;

b)  $\dot{\nabla}_k, \dot{\nabla}_k$  are the covariant  $h$ -derivative and covariant  $v$ -derivative with respect to this connection;

c)  $X_j^h, U_{pk}^q, V_{pk}^q$  are respectively arbitrary Finsler tensor fields, homogeneous of degree 1, 0 and  $-1$  with respect to  $y$ .

Among the set of Finsler almost symplectical connections (1.3) we point out two important.

The first one is the almost symplectical connection which results from (1.3) for  $X_j^h = 0, U_{pk}^q = 0, V_{pk}^q = 0$ . Since it generalizes Kawaguchi connection for the almost symplectical case, we are calling it Kawaguchi almost symplectical connection associated with the Finsler one  $F\dot{\Gamma}$  and it will be denoted by  $F\dot{\Gamma} = (\overset{k}{N}, \overset{k}{F}, \overset{k}{C})$ . Therefore, it is given by

$$(1.4) \quad \begin{cases} \overset{k}{N}_j^h = \dot{N}_j^h, \\ \overset{k}{F}_{jk}^h = \dot{F}_{jk}^h + \frac{1}{2} a^{ph} \dot{\nabla}_k a_{pj}, \\ \overset{k}{C}_{jk}^h = \dot{C}_{jk}^h + \frac{1}{2} a^{ph} \dot{\nabla}_k a_{pj}. \end{cases}$$

The second important almost symplectical connection is obtained from (1.3) if one considers as given connection  $F\dot{\Gamma}$ , the Matsumoto connection  $M\Gamma = (\overset{m}{N}, \overset{m}{F}, \overset{m}{C})$  and  $X_j^h = 0, U_{pk}^q = 0, V_{pk}^q = 0$ . We shall call it canonical almost symplectical connection and will be denoted by  $F\dot{\Gamma} = (\overset{q}{N}, \overset{q}{F}, \overset{q}{C})$ . It is expressed by

$$(1.5) \quad \begin{cases} \overset{q}{N}_j^h = \overset{m}{N}_j^h, \\ \overset{q}{F}_{jk}^h = \overset{m}{F}_{jk}^h + \frac{1}{2} a^{ph} \overset{m}{\nabla}_k a_{pj}, \\ \overset{q}{C}_{jk}^h = \overset{m}{C}_{jk}^h + \frac{1}{2} a^{ph} \overset{m}{\nabla}_k a_{pj}, \end{cases}$$

where  $M\Gamma = (\overset{m}{N}, \overset{m}{F}, \overset{m}{C})$  is Matsumoto connection



$$(1.6) \quad \begin{cases} \overset{m}{N}_j^h = \overset{\circ}{N}_j^h, \\ \overset{m}{F}_{jk}^h = \frac{\partial \overset{m}{N}_j^h}{\partial y^k}, \\ \overset{m}{C}_{jk}^h = \overset{\circ}{C}_{jk}^h, \end{cases}$$

with  $\overset{\circ}{N}_j^h$  — a nonlinear connection and  $\overset{\circ}{C}_{jk}^h$  — an arbitrary one, but fixed (in particular  $\overset{\circ}{C}_{jk}^h = 0$ ) and  $\overset{m}{\nabla}_k$ ,  $\overset{m}{V}_k$  represent the covariant  $h$ -derivative and covariant  $v$ -derivative with respect to this connection.

Now we put the following question : to get the class of Finsler connections  $F\overset{\circ}{\Gamma}$  having the property that Kawaguchi almost symplectical connection  $K\overset{\circ}{\Gamma}$  is the same with the canonical almost symplectical one  $F\overset{q}{\Gamma}$ . This class results from

**Theorem 1.2.** *All the Finsler connections  $F\overset{\circ}{\Gamma}$  which have the property  $K\overset{\circ}{\Gamma} = F\overset{q}{\Gamma}$  are given by*

$$(1.7) \quad \begin{cases} \overset{\circ}{N}_j^h = \overset{m}{N}_j^h = \overset{q}{N}_j^h, \\ \overset{\circ}{F}_{jk}^h = \overset{m}{F}_{jk}^h + \frac{1}{2} a^{ph} \overset{m}{V}_k + \theta_{mj}^{*hp} X_{pk}^m (= \overset{q}{F}_{jk}^h + \theta_{mj}^{*hp} X_{pk}^m), \\ \overset{\circ}{C}_{jk}^h = \overset{m}{C}_{jk}^h + \frac{1}{2} a^{ph} \overset{m}{V}_k + \theta_{mj}^{*hp} Y_{pk}^m (= \overset{q}{C}_{jk}^h + \theta_{mj}^{*hp} Y_{pk}^m), \end{cases}$$

where  $M\overset{\circ}{\Gamma} = (\overset{m}{N}, \overset{m}{F}, \overset{m}{C})$  is Matsumoto connection and  $X_{pk}^m$ ,  $Y_{pk}^m$  are arbitrary Finsler tensor fields.

**Proof.** Let be  $(A_j^h, B_{jk}^h, D_{jk}^h)$  the tensors obtained as difference of the pair  $(F\overset{\circ}{\Gamma}, F\overset{q}{\Gamma})$ :

$$(1.8) \quad \begin{cases} \overset{\circ}{N}_j^h = \overset{q}{N}_j^h + A_j^h, \\ \overset{\circ}{F}_{jk}^h = \overset{q}{F}_{jk}^h + B_{jk}^h, \\ \overset{\circ}{C}_{jk}^h = \overset{q}{C}_{jk}^h + D_{jk}^h. \end{cases}$$

Taking account of (1.8) in (1.4) we get

$$(1.9) \quad \begin{cases} \overset{k}{N}_j^h = \overset{q}{N}_j^h + A_j^h, \\ \overset{k}{F}_{jk}^h = \overset{q}{F}_{jk}^h + B_{jk}^h + \frac{1}{2} a^{ph} \left[ \overset{q}{V}_k a_{pj} - A_k^m \frac{\partial a_{pj}}{\partial y^m} - (B_{pk}^m a_{mj} + B_{jk}^m a_{pm}) \right], \\ \overset{k}{C}_{jk}^h = \overset{q}{C}_{jk}^h + D_{jk}^h + \frac{1}{2} a^{ph} \left[ \overset{q}{V}_k a_{pj} - (D_{pk}^m a_{mj} + D_{jk}^m a_{pm}) \right]. \end{cases}$$

But we have  $\nabla_k a_{pj} = 0$ ,  $\overset{q}{\nabla}_k a_{pj} = 0$ , and then (1.9) become

$$(1.10) \quad \begin{cases} N_j^h = \overset{q}{N}_j^h + A_j^h, \\ F_{jk}^h = \overset{q}{F}_{jk}^h - \frac{1}{2} a^{ph} A^m \frac{\partial a_{pj}}{\partial y^m} + A_{jm}^{ph} B_{pk}^m, \\ C_{jk}^h = \overset{q}{C}_{jk}^h + \theta_{jm}^{ph} D_{pk}^m. \end{cases}$$

By imposing  $K\overset{\circ}{\Gamma} = F\overset{\circ}{\Gamma}$ , the last equalities give

$$(1.11) \quad A_j^h = 0, \quad \theta_{jm}^{ph} B_{pk}^m = 0, \quad \theta_{jm}^p D_{pk}^m = 0.$$

The general solution of this system is

$$(1.12) \quad A_j^h = 0, \quad B_{jk}^i = \theta_{hj}^{*im} X_{mk}^h, \quad D_{jk}^i = \theta_{hj}^{*im} Y_{mk}^h,$$

with  $X_{mk}^h, Y_{mk}^h$  arbitrary tensor fields. Substituting in (1.8) and having in mind (1.5) we obtain (1.7).

2. Let be  $\mathcal{X}$  the set of Finsler connections  $F\overset{\circ}{\Gamma}$  with the property  $K\overset{\circ}{\Gamma} = F\overset{\circ}{\Gamma}$ .

**Theorem 2.1.** *The only connection  $F\overset{\circ}{\Gamma}$  from (1.7) which is almost symplectical, is the canonical one,  $F\overset{q}{\Gamma}$ .*

**Proof.** If the connection  $F\overset{\circ}{\Gamma}$  from (1.7) is almost symplectical, i.e. verifies (1.2), it follows

$$(2.1) \quad \begin{aligned} (a_{hj} \theta_{mi}^{*hp} + a_{ih} \theta_{mj}^{*hp}) X_{pk}^m &= 0, \\ (a_{hj} \theta_{mi}^{*hp} + a_{ih} \theta_{mj}^{*hp}) Y_{pk}^m &= 0. \end{aligned}$$

But these equations are equivalent to

$$(2.2) \quad \theta_{mi}^{*lp} X_{pk}^m = 0, \quad \theta_{mi}^{*lp} Y_{pk}^m = 0.$$

Taking into account (2.2), from (1.7) we get  $F\overset{\circ}{\Gamma} = F\overset{q}{\Gamma}$ .

**Theorem 2.2.** *All the connections  $F\overset{\circ}{\Gamma}$  from  $\mathcal{X}$  enjoy the following properties regarding the  $(h)h$ -torsion  $T$  and the  $(v)v$ -torsion  $S$ :*

$$(2.3) \quad a_{ih} \overset{\circ}{T}_{jk}^h + a_{jh} \overset{\circ}{T}_{ki}^h + a_{kh} \overset{\circ}{T}_{ij}^h = 0,$$

$$(2.4) \quad a_{ih} \overset{\circ}{S}_{jk}^h + a_{jh} \overset{\circ}{S}_{ki}^h + a_{kh} \overset{\circ}{S}_{ij}^h = 0.$$

The proof results by a direct computation of the left hand terms in (2.3), (2.4) and taking into account (1.7).



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STRUCTURI KAWAGUCHI ASOCIATE UNEI STRUCTURI FINSLER APROAPE  
SIMPLECTICE

(Rezumat)

Se determină clasa de conexiuni Finsler care are proprietatea că conexiunea aproape simplctică Kawaguchi coincide cu conexiunea aproape simplctică canonică.





# SUR LE THÉORÈME D'INVERSION LOCALE POUR DES APPLICATIONS ENTRE DES ESPACES LOCALEMENT CONVEXES

PAR

N. PAPAGHIUC

Le but de cette Note est de montrer que le théorème d'inversion locale connu pour le cas des espaces de Banach n'est pas, en général, vrai pour le cas des applications de classe  $C^1$  au sens de Marinescu entre des espaces localement convexes (voir même Fréchet). De cette cause, nous considérons une classe particulière des applications de classe  $C^1$  entre des espaces de Fréchet et généralisons le théorème d'inversion locale et le théorème de fonction implicite en utilisant seulement les topologies localement convexes des espaces de Fréchet.

Soit  $E$  un espace localement convexe réel et séparé Hausdorff et  $\Gamma_E$  l'ensemble de toutes les semi-normes continues sur  $E$ . Pour toute semi-norme  $|\cdot|_\alpha \in \Gamma_E$ , nous notons par  $N_\alpha = \{x \in E; |x|_\alpha = 0\}$ . Alors  $N_\alpha$  est un sous-espace vectoriel fermé de  $E$  et notons par  $E_\alpha = E/N_\alpha$  et par  $p_\alpha: E \rightarrow E_\alpha$  la projection canonique. Pour toute  $|\cdot|_\alpha \in \Gamma_E$ , l'espace  $E_\alpha$  est un espace vectoriel normé, la norme induite étant notée par  $\|\cdot\|_\alpha$  et donnée par  $\|p_\alpha(x)\|_\alpha = |x|_\alpha$ , pour tout  $p_\alpha(x) \in E_\alpha$ . On démontre facilement que le système  $\{E_\alpha, p_\alpha\}_{|\cdot|_\alpha \in \Gamma_E}$  est un système projectif d'espaces normés et que  $E = \lim_{|\cdot|_\alpha \in \Gamma_E} E_\alpha$ . Remarquons que si  $E$  est un espace de Fréchet-

Schwartz, alors  $E = \lim_{n \in \mathbb{N}} E_n$ , où chaque espace  $E_n$  est un espace de Banach.

Soient  $E$  et  $F$  deux espaces localement convexes,  $\Gamma_E$  et  $\Gamma_F$  les ensembles de toutes les semi-normes continues sur  $E$  et respectivement  $F$  et soit  $\varphi: \Gamma_F \rightarrow \Gamma_E$  une application donnée. Nous désignons par  $L(E, F)$  l'espace vectoriel de toutes les applications linéaires et continues de  $E$  dans  $F$  et par  $L_\varphi(E, F)$  l'espace localement convexe de toutes les applications linéaires de  $E$  dans  $F$ , continues par rapport à l'application  $\varphi$ , c'est-à-dire :  $L_\varphi(E, F) =$

$$= \left\{ u \in L(E, F) : |x|_{\varphi(\beta)} = 0 \text{ implique } |u(x)|_\beta = 0 \text{ et } |u|_{\beta, \varphi(\beta)} = \right.$$

$$\left. = \sup_{|x|_{\varphi(\beta)} \neq 0} \frac{|u(x)|_\beta}{|x|_{\varphi(\beta)}} < \infty \text{ pour toute } |\cdot|_\beta \in \Gamma_F \right\} \text{ (voir [4, p. 99]). Alors } L(E, F) =$$

$$= \bigcup_{\varphi \in \Phi} L_\varphi(E, F), \text{ où par } \Phi \text{ on a noté l'ensemble de toutes les applications } \varphi: \Gamma_F \rightarrow \Gamma_E.$$



Soit  $U \subset E$  un sous-ensemble ouvert de  $E$ ,  $\varphi: \Gamma_F \rightarrow \Gamma_E$  une application donnée et  $f: U \rightarrow F$  une application différentiable sur  $U$  par rapport à  $L_\varphi(E, F)$  au sens de la définition 1 de [5, p. 10].

Nous dirons que  $f$  est de classe  $C^1$  sur  $U$  si l'application dérivée  $f': U \rightarrow L_\varphi(E, F)$  est aussi continue. Supposons que l'application  $\varphi$  est bijective et que le sous-ensemble  $f(U)$  est aussi ouvert dans  $F$ . Alors, nous dirons que  $f: U \rightarrow f(U)$  est un difféomorphisme de classe  $C^1$  par rapport à  $L_\varphi(E, F)$  si  $f$  est une application bijective,  $f$  est de classe  $C^1$  par rapport à  $L_\varphi(E, F)$  et  $f^{-1}: f(U) \rightarrow U$  est de classe  $C^1$  par rapport à  $L_{\varphi^{-1}}(F, E)$ .

On sait que, si  $E$  et  $F$  sont deux espaces localement convexes (voir même Fréchet) et  $\varphi: \Gamma_F \rightarrow \Gamma_E$  est une application bijective, alors l'ensemble  $\text{Isom}_\varphi(E, F) = \text{Isom}(E, F) \cap L_\varphi(E, F)$  n'est pas, en général ouvert dans  $L_\varphi(E, F)$  et par suite le théorème d'inversion locale connu pour le cas des espaces de Banach n'est pas, en général vrai pour le cas des applications de classe  $C^1$  entre deux espaces localement convexes. Ce fait il résulte immédiatement en considérant l'exemple suivant.

*Exemple.* Soit  $E = R^\infty = \lim_{n \in \mathbb{N}} R^n$  qui est un espace de Fréchet et soit  $f: R^\infty \rightarrow R^\infty$  l'application définie par  $f(x_1, x_2, \dots, x_n, \dots) = (x_1 - x_1^2, x_2 - x_2^2, \dots, x_n - x_n^2, \dots)$ . On vérifie facilement que l'application  $f$  est de classe  $C^1$  par rapport à  $L_{\varphi_0}(R^\infty, R^\infty)$ , où  $\varphi_0: \Gamma_{R^\infty} \rightarrow \Gamma_{R^\infty}$  est l'application identique et que  $f'(0) = I_{R^\infty}$ , mais n'existe aucun voisinage ouvert  $U$  de  $0 \in R^\infty$  telle  $f: U \rightarrow f(U)$  soit un difféomorphisme de classe  $C^1$ . En effet, si nous considérons la suite  $(x^k)_{k \in \mathbb{N}}$  donné par  $x^k = (\delta^k)$ , où  $\delta_i^k = 1$  pour  $i = k$  et  $\delta_i^k = 0$  pour  $i \neq k$ , nous avons que pour tout voisinage ouvert  $U$  de  $0$  dans  $R^\infty$  il existe un  $k_0 \in \mathbb{N}$  tel que  $k > k_0$  implique  $x^k \in U$ . Parce que  $f(x^k) = 0$  pour tout  $k \in \mathbb{N}$ , il résulte que pour tout voisinage ouvert  $U$  de  $0$  dans  $R^\infty$  la restriction  $f: U \rightarrow f(U)$  n'est pas injective et par suite n'est pas un difféomorphisme. Remarquons qu'une généralisation du théorème d'inversion locale pour des applications de classe  $C^1$  entre deux espaces de Fréchet a été donné dans [5], mais en utilisant essentiellement les topologies raffinées des espaces de Fréchet (voir aussi [6]).

Dans ce qu'il suit, nous considérons une classe particulière des applications de classe  $C^1$  entre des espaces de Fréchet et généralisons le théorème d'inversion locale et le théorème de fonction implicite en utilisant seulement les topologies localement convexes. Dans ce but, nous supposons que  $E$  et  $F$  sont deux espaces de Fréchet et que  $\varphi: \Gamma_F \rightarrow \Gamma_E$  est une application bijective.

**L e m m e 1.** Soit  $u \in \text{Isom}_\varphi(E, F)$ . Alors, pour toute semi-norme  $|\cdot|_\alpha \in \Gamma_E$ , l'application induite  $u_\alpha: E_\alpha \rightarrow F_{\varphi^{-1}(\alpha)}$  définie par  $u_\alpha(p_\alpha(x)) = p_{\varphi^{-1}(\alpha)}(u(x))$  est un isomorphisme des espaces normés, c'est-à-dire  $u_\alpha \in \text{Isom}(E_\alpha, F_{\varphi^{-1}(\alpha)})$ .

La démonstration du Lemme 1 est immédiate, tenant compte de la définition de  $\text{Isom}_\varphi(E, F)$  et de la commutativité de la diagramme suivante



$$\begin{array}{ccc} & u & \\ E & \xrightarrow{\quad} & F \\ p_\alpha \downarrow & & \downarrow p_{\varphi^{-1}(\alpha)} \\ E_\alpha & \xrightarrow{u_\alpha} & F_{\varphi^{-1}(\alpha)} \end{array}$$

**L e m m e 2.** Soit  $u \in L_\varphi(E, F)$ , l'application  $\varphi$  étant bijective et supposons qu'il existe une semi-norme  $|\cdot|_\varphi \in \Gamma_E$  telle que l'application induite  $u_\alpha: E_\alpha \rightarrow F_{\varphi^{-1}(\alpha)}$  soit isomorphisme. Nous supposons aussi que la restriction  $u|_{N_\alpha}: N_\alpha \rightarrow N_{\varphi^{-1}(\alpha)}$  coïncide avec la restriction à  $N_\alpha$  d'un isomorphisme  $u_0 \in \text{Isom}_\varphi(E, F)$ , c'est-à-dire  $u(h) = u_0(h)$  pour tout  $h \in N_\alpha$ . Alors  $u \in \text{Isom}_\varphi(E, F)$ .

**D é m o n s t r a t i o n.** Nous notons par  $\beta = \varphi^{-1}(\alpha)$  et alors nous avons que  $u_\alpha(p_\alpha(x)) = p_\beta(u(x))$  pour tout  $x \in E$ . Parce que les espaces  $E$  et  $F$  sont des espaces de Fréchet, il est suffisant de montrer que l'application  $u$  est bijective.

Soit  $x_1 \neq x_2$ . Montrons que  $u(x_1) \neq u(x_2)$ . En supposant que  $x_1, x_2 \in N_\alpha$  il résulte que  $u(x_1) - u(x_2) = u_0(x_1 - x_2) \neq 0$ , parce que  $x_1 - x_2 \neq 0$  et  $u_0 \in \text{Isom}_\varphi(E, F)$  et donc, dans ce cas  $u(x_1) \neq u(x_2)$ . Si  $x_1 \in N_\alpha$  et  $x_2 \notin N_\alpha$  alors  $u(x_1) = u_0(x_1) \in N_\beta$  mais  $u(x_2) \notin N_\beta$  et par suite il résulte aussi  $u(x_1) \neq u(x_2)$ . En effet, la relation  $u_\alpha(p_\alpha(x_2)) = p_\beta(u(x_2))$  et le fait que  $p_\alpha(x_2) \neq 0$  et  $u_\alpha \in \text{Isom}(E_\alpha, F_\beta)$  impliquent  $p_\beta(u(x_2)) \neq 0$ , c'est-à-dire  $u(x_2) \notin N_\beta$ . Supposons maintenant que  $x_1, x_2 \notin N_\alpha$  et  $x_1 \neq x_2$ . Alors  $x_1 - x_2 \in N_\alpha$  où  $x_1 - x_2 \notin N_\alpha$ . Si  $x_1 - x_2 \in N_\alpha$  nous avons que  $u(x_1 - x_2) = u_0(x_1 - x_2) \neq 0$  et donc  $u(x_1) \neq u(x_2)$ . Si  $x_1 - x_2 \notin N_\alpha$  alors  $p_\alpha(x_1) \neq p_\alpha(x_2)$  et parce que  $u_\alpha \in \text{Isom}(E_\alpha, F_\beta)$  il résulte que  $u_\alpha(p_\alpha(x_1)) \neq u_\alpha(p_\alpha(x_2))$ . La dernière relation implique  $p_\beta(u(x_1)) \neq p_\beta(u(x_2))$ , c'est-à-dire  $u(x_1) - u(x_2) \notin N_\beta$  et par suite  $u(x_1) \neq u(x_2)$ .

Montrons que l'application  $u$  est aussi surjective. Soit  $y \in F$ . Si  $y \in N_\beta$  alors il existe  $x \in N_\alpha$  tel que  $u_0(x) = y$  et donc  $u(x) = y$ . Supposons que  $y \notin N_\beta$ . Alors  $p_\beta(y) \neq 0$  et par suite il existe un élément  $p_\alpha(x) \in E_\alpha$ ,  $p_\alpha(x) \neq 0$ , tel que  $u_\alpha(p_\alpha(x)) = p_\beta(y)$ . Parce que  $u_\alpha(p_\alpha(x)) = p_\beta(u(x))$  il résulte qu'il existe  $x \in E$  tel que  $p_\beta(u(x)) = p_\beta(y)$ , c'est-à-dire tel que  $u(x) - y \in N_\beta$ . Nous obtenons alors qu'il existe un élément  $x_0 \in N_\alpha$  tel que  $u_0(x_0) = u(x) - y$ . La dernière relation est équivalente avec  $y = u(x - x_0)$  et par suite nous avons que l'application  $u$  est aussi surjective.

Soient maintenant  $E$  et  $F$  deux espaces de Fréchet et  $\varphi: \Gamma_F \rightarrow \Gamma_E$  une application bijective. Supposons qu'il existe une semi-norme  $|\cdot|_\beta \in \Gamma_F$  tels que les espaces normés  $E_{\varphi(\beta)}$  et  $F_\beta$  soient des espaces de Banach. Soit  $U$  un sous-ensemble ouvert et convexe de  $E$  tel que  $U = p_{\varphi(\beta)}^{-1}(U_{\varphi(\beta)})$ , où  $U_{\varphi(\beta)}$  est un sous-ensemble ouvert de  $E_{\varphi(\beta)}$  et soit  $f: U \rightarrow F$  une application de classe  $C^1$  par rapport à  $L_\varphi(E, F)$ . Supposons aussi qu'il existe un point  $x_0 \in U$  telle que  $f'(x_0) \in \text{Isom}_\varphi(E, F)$  et que pour tout  $x \in U$ , a lieu la relation  $f'(x)(h) = f'(x_0)(h)$  pour tout  $h \in N_{\varphi(\beta)}$ , c'est-à-dire  $f'(x)|_{N_{\varphi(\beta)}} = f'(x_0)|_{N_{\varphi(\beta)}}$  pour tout  $x \in U$ .

**T h é o r è m e 1** (d'inversion locale). Si  $f$  satisfait les conditions ci-dessus, alors il existe un voisinage ouvert  $U_0$  de  $x_0$  dans  $E$ ,  $U_0 \subset U$  et un voisinage ouvert  $V_0$  de  $f_1(x_0)$  dans  $F$  telle que  $f: U_0 \rightarrow V_0$  soit un difféomorphisme de classe  $C^1$  par rapport à  $L_\varphi(E, F)$ .



**Démonstration.** Nous notons  $\alpha = \varphi(\beta) \in \Gamma_E$ . Alors, si  $f$  satisfait les conditions du Théorème 1, le Lemme 1 implique que l'application induite  $f_\alpha: U_\alpha \rightarrow F_\beta$  définie par  $f_\alpha(p_\alpha(x)) = p_\alpha(f(x))$  satisfait les conditions du théorème d'inversion locale pour le cas des espaces de Banach au point  $p_\alpha(x_0) \in U_\alpha$ . Il résulte alors qu'il existe un voisinage ouvert  $U_\alpha^0$  de  $p_\alpha(x_0)$  dans  $E_\alpha$ ,  $U_\alpha^0 \subset U_\alpha$  et un voisinage ouvert  $V_\beta^0$  de  $f_\alpha(p_\alpha(x_0))$  dans  $F_\beta$  telle que l'application  $f_\alpha: U_\alpha^0 \rightarrow V_\beta^0$  soit un difféomorphisme de classe  $C^1$ . Notons par  $U_0 = p_\alpha^{-1}(U_\alpha^0) \subset U$  et par  $V_0 = p_\beta^{-1}(V_\beta^0)$  et montrons que l'application  $f: U_0 \rightarrow V_0$  est un difféomorphisme de classe  $C^1$  par rapport à  $L_\varphi(E, F)$ . Remarquons que, en utilisant le Lemme 2, il résulte que  $f'(x) \in \text{Isom}_\varphi(E, F)$  pour tout  $x \in U$ . Montrons que l'application  $f: U_0 \rightarrow V_0$  est bijective.

Soit  $x_1, x_2 \in U_0$ ,  $x_1 \neq x_2$ . Si  $p_\alpha(x_1) \neq p_\alpha(x_2)$  alors  $f_\alpha(p_\alpha(x_1)) \neq f_\alpha(p_\alpha(x_2))$  donc  $p_\beta(f(x_1)) \neq p_\beta(f(x_2))$  et par suite  $f(x_1) \neq f(x_2)$ . Supposons que  $x_1 \neq x_2$  mais  $p_\alpha(x_1) = p_\alpha(x_2)$ . Alors, le théorème des accroissements finis [5, p. 16] implique  $f(x_1) - f(x_2) = f'(c)(x_1 - x_2)$  où  $c \in [x_1, x_2] \subset U_0$  et parce que  $x_1 - x_2 \in N_\alpha$  nous avons que  $f'(c)(x_1 - x_2) = f'(x_0)(x_1 - x_2)$  par suite  $f(x_1) - f(x_2) = f'(x_0)(x_1 - x_2)$ . Tenant compte que  $x_1 - x_2 \neq 0$  et  $f'(x_0) \in \text{Isom}_\varphi(E, F)$  il résulte que  $f(x_1) \neq f(x_2)$  et donc que l'application  $f: U_0 \rightarrow V_0$  est injective. Soit maintenant  $y \in V_0$ . Alors  $p_\beta(y) \in V_\beta^0$  et alors il existe un élément  $p_\alpha(x) \in U_\alpha^0$  telle que  $f_\alpha(p_\alpha(x)) = p_\beta(y)$ . Parce que  $f_\alpha(p_\alpha(x)) = p_\beta(f(x))$  il résulte que  $p_\beta(y) = p_\beta(f(x))$ , c'est-à-dire  $y - f(x) \in N_\beta$ . Nous obtenons alors qu'il existe un élément  $h \in N_\alpha$  telle que  $f'(x_0)(h) = y - f(x)$  qui implique  $y = f(x + h)$  et par suite que l'application  $f: U_0 \rightarrow V_0$  est aussi surjective. La proposition 6 de [5, p. 28] implique alors que l'application  $f: U_0 \rightarrow V_0$  est un difféomorphisme de classe  $C^1$  par rapport à  $L_\varphi(E, F)$ .

D'une manière classique, le Théorème 1 implique le

**Théorème 2** (de fonction implicite). Soient  $E, F$  et  $G$  trois espaces de Fréchet et  $\varphi: \Gamma_G \rightarrow \Gamma_E$  et  $\psi: \Gamma_G \rightarrow \Gamma_F$  deux applications telles que l'application  $\psi$  soit bijective. Supposons qu'il existe une semi-norme  $|\cdot|_\gamma \in \Gamma_G$  tels que les espaces  $F_{\psi(\gamma)}$  et  $G_\gamma$  soient des espaces de Banach et soit  $U \times V$  un sous-ensemble ouvert de  $E \times F$ , où  $V = p_{\psi(\gamma)}^{-1}(V_{\psi(\gamma)})$ ,  $V_{\psi(\gamma)}$  étant un sous-ensemble ouvert de  $F_{\psi(\gamma)}$ . Soit  $f: U \times V \rightarrow G$  une application de classe  $C^1$  par rapport à  $L_{\varphi(\psi)}(E \times F, G)$ , où  $(\varphi, \psi): \Gamma_G \rightarrow \Gamma_E \times \Gamma_F$  est définie par  $(\varphi, \psi)(\gamma) = (\varphi(\gamma), \psi(\gamma))$ .

Supposons aussi que l'application  $f$  satisfait les trois conditions suivantes :

- (1) il existe  $(x_0, y_0) \in U \times V$  tel que  $f(x_0, y_0) = 0$ ;
- (2)  $f'_y(x_0, y_0) \in \text{Isom}_\psi(F, G)$ ;
- (3) pour tout  $(x, y) \in U \times V$ , a lieu la relation  $f'_y(x, y)(k) = f'_y(x_0, y_0)(k)$ , pour tout  $k \in N_{\psi(\gamma)}$ .

Alors il existe un voisinage ouvert  $U_0 \times V_0$  de  $(x_0, y_0)$ ,  $U_0 \times V_0 \subset U \times V$  un voisinage ouvert  $U_1$  de  $x_0$  et une application  $g: U_1 \rightarrow F$  de classe  $C^1$  telle que la relation  $(x, y) \in U_0 \times V_0$  et  $f(x, y) = 0$  soit équivalente avec  $x \in U_1$  et  $y = g(x)$ .



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## ASUPRA TEOREMEI DE INVERSARE LOCALĂ PENTRU APLICAȚII ÎNTRE SPAȚII LOCAL-CONVEXE

(Rezumat)

Se pune în evidență faptul că, cunoscuta teoremă de inversare locală pentru aplicații de clasă  $C^1$  între două spații Banach nu poate fi extinsă, în general, pentru aplicații de clasă  $C^1$  între două spații local-convexe, chiar dacă spațiile local-convexe considerate sînt spații Fréchet. Se consideră atunci o clasă particulară de aplicații de clasă  $C^1$  în sens Marinescu între spații Fréchet pentru care se generalizează teorema de inversare locală și teorema funcției implicite, utilizînd numai topologiile local-convexe ale spațiilor Fréchet.

The first part of the document discusses the importance of maintaining accurate records of all transactions. It emphasizes that proper record-keeping is essential for the integrity of the financial system and for the ability to detect and prevent fraud. The document also outlines the responsibilities of individuals involved in the process, including the need for transparency and accountability.

### II. Objectives and Scope

The primary objective of this study is to evaluate the effectiveness of the current record-keeping system and to identify areas for improvement. The scope of the study is limited to the financial transactions of the organization over a period of six months. The study will involve a thorough review of all records, as well as interviews with key personnel involved in the process.



## ALGORITM ȘI PROGRAM FORTRAN PENTRU REZOLVAREA SISTEMELOR ALGEBRICE NELINIARE

DE

NINA POEATĂ și NONEL THIRER

**1. Introducere.** Analiza diverselor tipuri de sisteme electrotehnice sau mecanice reale conduce la sisteme de ecuații algebrice neliniare, pentru rezolvarea cărora însă nu există subprograme în bibliotecile matematice consultate de noi (ale calculatoarelor IRIS-50, IBM-360).

În lucrare se prezintă algoritmul de calcul și programul FORTRAN pentru rezolvarea sistemelor algebrice neliniare, folosind metoda gradientului.

**2. Metoda gradientului de rezolvare a unui sistem algebric neliniar.** Se dă sistemul algebric neliniar de forma

$$(1) \quad [f([x])] = 0,$$

în care

$$(2) \quad [x] = x_1, x_2, \dots, x_n$$

și

$$(3) \quad [f([x])] = \begin{bmatrix} f_1([x]) = 0 \\ f_2([x]) = 0 \\ \dots \\ f_n([x]) = 0 \end{bmatrix}.$$

În ipoteza că funcțiile  $f_i([x])$  sînt reale și continuu diferențiabile în domeniul lor comun de definiție (condiție îndeplinită de orice sistem de ecuații care descrie funcționarea în regim stabil a unui sistem tehnic real) și că există o soluție unică în acest domeniu, se știe [1] că respectiva soluție reprezintă minimul strict al funcției potențial  $u([x])$  definită cu expresia

$$(4) \quad u([x]) = \sum_{j=1}^n (f_j([x]))^2.$$

În cele ce urmează, găsirea soluției sistemului (1) este înlocuită prin căutarea minimului strict al expresiei (4) în respectivul domeniu. În acest scop se folosește un proces iterativ de calcul și anume

$$(5) \quad [x]^{(i+1)} = [x]^{(1)} - K_i [\text{grad } u([x])]^{(i)},$$

în care

$$(6) \quad [x]^{(i+1)} = x_1^{(i+1)}, x_2^{(i+1)}, \dots, x_n^{(i+1)}$$

sînt soluțiile obținute în iterația  $i + 1$ ,

$$(7) \quad [x]^{(i)} = x_1^{(i)}, x_2^{(i)}, \dots, x_n^{(i)}$$

sînt soluțiile obținute în iterația precedentă  $i$ ,

$$(8) \quad [\text{grad } u([x])]^{(i)} = \begin{bmatrix} \left( \frac{\partial u([x])}{\partial x_1} \right)^{(i)} \\ \dots \\ \left( \frac{\partial u([x])}{\partial x_n} \right)^{(i)} \end{bmatrix}$$

este gradientul funcției potențial în punctul  $[x]^{(i)}$ .

Pentru stabilirea expresiei de calcul a factorului  $K_i$  se consideră funcția potențial (4) în punctul  $[x]^{(i+1)}$ , scoțind în evidență variabila  $K_i$ . Notăm această funcție scalară cu  $\Phi(K_i)$ . Expresia ei este

$$(9) \quad \Phi([K_i]) = \sum_{j=1}^n \{f_j([x]^{(i)} - K_i [\text{grad } u([x])]^{(i)})\}^2.$$

Din condiția

$$(10) \quad \Phi([K_i]) = \text{minim}$$

rezultă

$$(11) \quad \frac{d}{dK_i} \sum_{j=1}^n \{f_j([x]^{(i)} - K_i [\text{grad } u([x])]^{(i)})\}^2 = 0.$$

În ipoteza  $K_i$  mic, reținînd numai primii doi termeni din dezvoltarea în serie Taylor a termenilor de sumat, rezultă că în (11) termenul  $j$  este de forma

$$(12) \quad \begin{aligned} f_j([x]^{(i)} - K_i [\text{grad } u([x])]^{(i)}) &\cong \\ &\cong f_j([x]^{(i)}) - K_i f'_j([x]^{(i)}) [\text{grad } u([x])]^{(i)}, \end{aligned}$$

în care

$$(13) \quad f'_j([x]) = \left\{ \left( \frac{\partial f_j([x])}{\partial x_1} \right)^{(i)}, \dots, \left( \frac{\partial f_j([x])}{\partial x_n} \right)^{(i)} \right\}$$

și constituie linia  $j$  a jacobianului funcțiilor  $[f([x]^{(i)})]$ .

Introducînd (12) în (11) se obține după derivare

$$(14) \quad -2 \sum_{j=1}^n [f_j([x]^{(i)}) - K_i f'_j([x]^{(i)}) [\text{grad } u([x])^{(i)}] f'_j([x]^{(i)}) [\text{grad } u([x])^{(i)}] = 0,$$

din care rezultă expresia de calcul pentru  $K_i$  și anume



$$(15) \quad K_i = \frac{\sum_{j=1}^n f_j([x]^{(t)}) \{f'_j([x])^{(t)} [\text{grad } u([x])]^{(t)}\}}{\sum_{j=1}^n \{f'_j([x])^{(t)} [\text{grad } u([x])]^{(t)}\}^2}.$$

Pe de altă parte, o componentă  $k$  a vectorului  $[\text{grad } u([x])]^{(t)}$  este de forma

$$(16) \quad \left[ \frac{\partial u([x])}{\partial x_k} \right]^{(t)} = \left[ \frac{\partial}{\partial x_k} \sum_{j=1}^n (f_j([x])^2) \right]^{(t)} = 2 \sum_{j=1}^n f_j([x]^{(t)}) \left\{ \frac{\partial f_j([x])}{\partial x_k} \right\}^{(t)}.$$

Rezultă deci

$$(17) \quad [\text{grad } u([x])]^{(t)} = 2 \begin{bmatrix} f_1([x]^{(t)}) \left\{ \frac{\partial f_1([x])}{\partial x_1} \right\}^{(t)} + \dots + f_n([x]^{(t)}) \left\{ \frac{\partial f_n([x])}{\partial x_1} \right\}^{(t)} \\ \dots \\ f_1([x]^{(t)}) \left\{ \frac{\partial f_1([x])}{\partial x_n} \right\}^{(t)} + \dots + f_n([x]^{(t)}) \left\{ \frac{\partial f_n([x])}{\partial x_n} \right\}^{(t)} \end{bmatrix}.$$

Ținînd cont de (13) și (17) obținem

$$(18) \quad f'_j([x])^{(t)} [\text{grad } u([x])]^{(t)} = \sum_{k=1}^n \frac{\partial f_j([x])^{(t)}}{\partial x_k} 2 \left( \sum_{i=1}^n f_i([x]^{(t)}) \left\{ \frac{\partial f_i([x])}{\partial x_k} \right\}^{(t)} \right).$$

Introducînd (18) în (15) se obține în final următoarea expresie de calcul numeric pentru  $K_i$ :

$$(19) \quad K_i = \frac{\sum_{j=1}^n f_j([x]^{(t)}) \left( \sum_{k=1}^n \left\{ \frac{\partial f_j([x])}{\partial x_k} \right\}^{(t)} 2 \left( \sum_{i=1}^n f_i([x]^{(t)}) \left\{ \frac{\partial f_i([x])}{\partial x_k} \right\}^{(t)} \right) \right)}{\sum_{j=1}^n \left( \sum_{k=1}^n \left\{ \frac{\partial f_j([x])}{\partial x_k} \right\}^{(t)} 2 \left( \sum_{i=1}^n f_i([x]^{(t)}) \left\{ \frac{\partial f_i([x])}{\partial x_k} \right\}^{(t)} \right) \right)^2}.$$

Condiția de convergență a procesului de calcul este dată de relația

$$(20) \quad M([x]^{(t+1)}) < M([x]^{(t)}).$$

**3. Algoritmul de calcul.** Pe baza metodei de calcul descrisă în paragraful precedent, s-a stabilit algoritmul numeric de calcul al sistemului algebric neliniar. În fig. 1 se dă o schemă bloc de calcul a problemei în care s-au folosit următoarele denumiri de variabile, păstrate și în program:  $X$  — tabelul monodimensional al necunoscutelor; programatorul îi atribuie valori inițiale iar programul furnizează soluțiile corecte;  $EPS$  — eroarea admisă de programator;  $ITMAX$  — numărul maxim de iterații, ales inițial de programator;  $N$  — numărul de ecuații;  $F$  — tabelul monodimensional pentru valorile numerice ale funcțiilor date;  $W$  — matricea pentru valorile numerice ale tuturor derivatelor parțiale (jacobianul);  $GDU$  — tabel monodimensional pentru valori intermediare de calcul (expresia (17));  $CALCUL$  — denumirea subprogramului în care se face cal-

culul valorilor numerice pentru funcții și derivate, apelat în subprogramul **GRADIENT** care rezolvă sistemul neliniar; **KOD** — codul de eroare al cărui valoare se precizează în subprogramul **GRADIENT** și are următoarele semnificații: **KOD** = 0 semnifică obținerea corectă a soluțiilor în limitele erorii admise inițial de programator (**EPS**), **KOD** = 1 semnifică

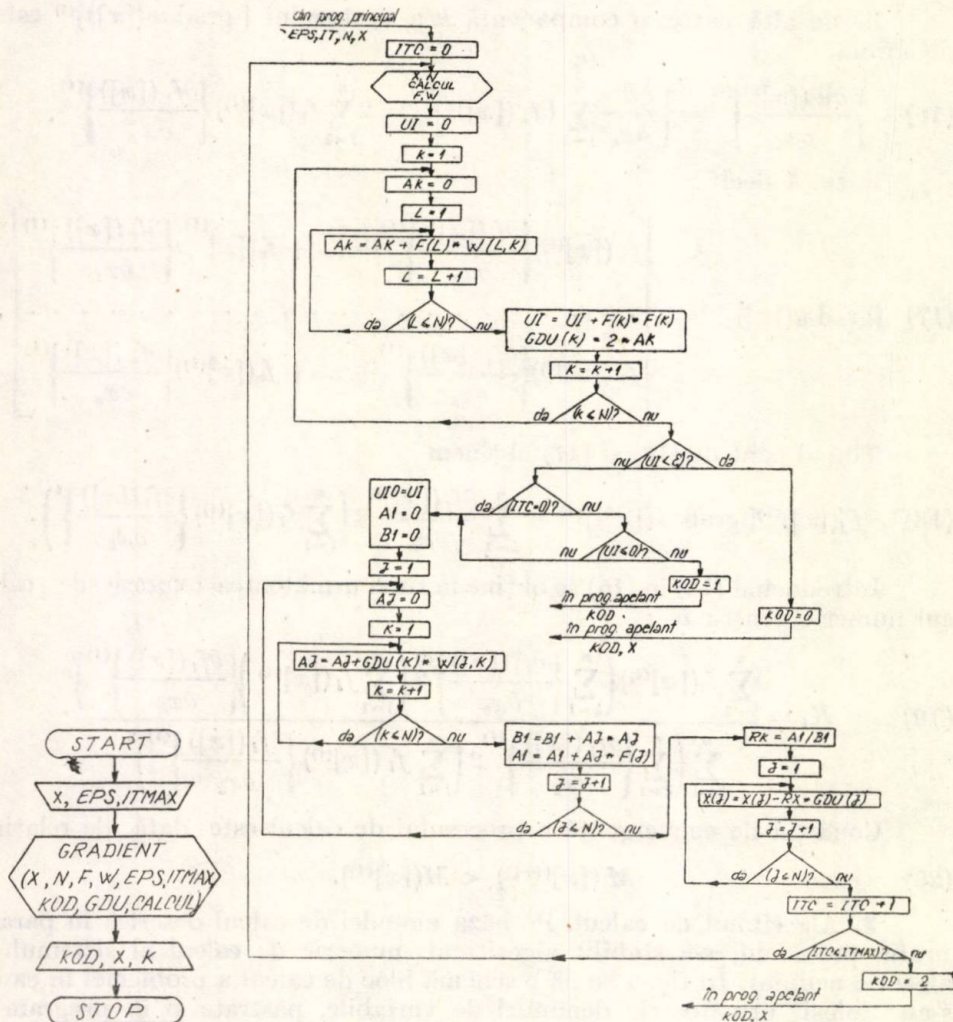


Fig. 1

Fig. 2

neconvergența procesului de calcul (nu e satisfăcută condiția (20));  $K$  — numărul efectiv de iterații efectuat în scopul obținerii rezultatelor în condițiile inițiale impuse.

În fig. 2 se dă organigrama de calcul detaliată a subprogramului **GRADIENT** iar în fig. 3 se prezintă subprogramul, valabil pentru orice sistem neliniar.



```

1  SUBROUTINE GRADIENT(X,N,P,W,EPS,ITMAX, KOD, GDU, CALCUL)
2  DIMENSION X(N),F(N),W(N,N),GDU(N)
3  ITC=0
4  CALL CALCUL(X,N,F,W)
5  UI=0.
6  DO 14 K=1,N
7    AK=0.
8    DO 12 L=1,N
9    12 AK=AK+F(L)*W(L,K)
10   UI=UI+F(K)*F(K)
11  14 GDU(K)=2*AK
12   IF(UI.LT.EPS)GOTO 6
13   IF(ITC)2,3,2
14   2 IF(UI--UIO)3,3,4
15   4 KOD=1
16   RETURN
17   6 KOD=0
18   ITMAX=ITC
19   RETURN
20   3 UIO=UI
21   A1 = 0.
22   B1 = 0.
23   DO 10 J = 1,N
24     AJ = 0.
25     DO 11 K = 1,N
26     11 AJ = AJ + GDU(K)* W(J,K)
27     B1 = B1 + AJ*AJ
28     10 A1 = A1 + AJ * F(J)
29     RK = A1/B1
30     DO 13 J = 1,N
31     13 X(J) = X(J)-RK* GDU(J)
32     ITC = ITC + 1
33     IF(ITC.LE.ITMAX)GO TO 20
34     KOD=2
35     RETURN
36   END

```

FORTRAN 00.00

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Fig. 3

```

1  SUBROUTINE CALCUL(X,N,F,W)
2  DIMENSION X(N),F(N),W(N,N)
3  F(1)=X(1)+X(1)*X(1)-2*X(2)*X(3)-0.1
4  F(2)=X(2)-X(2)*X(2)+3*X(1)*X(3)+0.2
5  F(3)=X(3)+X(3)*X(3)+2*X(1)*X(2)-0.3
6  W(1,1)=1+2*X(1)
7  W(1,2)=-2*X(3)
8  W(1,3)=-2*X(2)
9  W(2,1)=3*X(3)
10  W(2,2)=1-2*X(2)
11  W(2,3)=3*X(1)
12  W(3,1)=2*X(2)
13  W(3,2)=2*X(1)
14  W(3,3)=1+2*X(3)
15  RETURN
16  END

```

FORTRAN 00.00

GRADLN 15/09/78 12.57.17

```

1  REAL X(3),F(3),W(3,3),R(3)
2  EXTERNAL CALCUL,TERMIST
3  3 READ(105,1,END=10)X,EPS,K
4  1 FORMAT(4F10.6,13)
5  CALL GRADIENT (X,3,F,W,EPS,K,KOD,R,CALCUL)
6  PRINT 2,KOD,X,EPS,K
7  2 FORMAT(5X,KOD='11,5X,SOLUTII'3D15.7,5X,' EPS=' 'D15.7,5X'K=' 'I8)
8  GO TO 3
9  10 STOP
10  END

```

FORTRAN 00.00

GRADLN 15/09/78 12.57.17

Fig. 4



|       |         |              |                |              |                    |        |
|-------|---------|--------------|----------------|--------------|--------------------|--------|
| KOD=0 | SOLUTII | .1780736D-01 | -1.1767717D+00 | .2453675D+00 | EPS = .9999999D-03 | K = 3  |
| KOD=0 | SOLUTII | .1282857D-01 | -1.1778059D+00 | .2446900D+00 | EPS = .1000000D-08 | K = 8  |
| KOD=1 | SOLUTII | .1282417D-01 | -1.1778007D+00 | .2446881D+00 | EPS = .9999999D-15 | K = 25 |

\*STOP\*

Fig. 5

```

1  SUBROUTINE TERMIST(X,N,F,W)
2  DIMENSION X(N),F(N),W(N,N)
3  DATA R0,R2,R3,R4,R5,R6,B,TO,H,ETA,RH/4*1000.,10.,1.,3000.,293.,0.0.
4  E=4.5
5  C1=.8,100./
6  F(1)=X(1)-X(2)-X(3)
7  F(2)=X(3)-X(4)-X(5)
8  F(3)=X(6)-X(1)-X(4)
9  T1=TO+ETA*RH*X(1)**2/H
10 R1=R0*EXP(B/T1-B/TO)
11 F(4)=R1*X(1)+R5*X(5)-R4*X(4)
12 F(5)=R2*X(2)-R3*X(3)-R5*X(5)
13 F(6)=R6*X(6)+R1*X(1)+R2*X(2)-E
14 DO 6 I=1,6
15   DO 6 J=1,6
16    W(I,J)=0.
17    W(1,1)=0.
18    W(1,2)=-1.
19    W(1,5)=-1.
20    W(2,3)=1.
21    W(2,4)=-1.
22    W(2,5)=-1.
23    W(3,1)=-1.
24    W(3,4)=-1.
25    W(3,6)=1.
26    W(4,1)=R1-R1*X(1)**2*R/T1**2*ETA2RH/H
27    W(4,5)=R5
28    W(4,4)=-R4
29    W(5,2)=R2
30    W(5,3)=-R3
31    W(5,5)=-R5
32    W(6,1)=W(4,1)
33    W(6,2)=R2
34    W(6,6)=R6
35  RETURN
36  END

```

FORTRAN 00.00

GRAD 28/09/78 11.37.08

Fig. 6

|        |              |              |              |               |              |              |              |
|--------|--------------|--------------|--------------|---------------|--------------|--------------|--------------|
| KOD=0  | SOLUTII      | .2242893D-02 | .2242205D-02 | .2242297D-02  | .2239760D-02 | .1456101D-07 | .4507880D-05 |
| EPS=   | .9999999D-03 | K=10         |              |               |              |              |              |
| *STOP* |              |              |              |               |              |              |              |
| 0053   | GRAD         | AN=23A0      | PH=0008      | DATE=23/09/78 |              |              |              |

Fig. 7



4. **Exemple de folosire a subprogramului.** În fig. 4 se dă programul principal și subprogramul *CALCUL* pentru un sistem neliniar cu 3 ecuații și 3 necunoscute. Prin folosirea instrucțiunii *READ* cu opțiunea *END* în programul principal, sistemul poate fi soluționat pentru orice număr de valori inițiale. În fig. 5 se prezintă rezultatele obținute pentru următoarele 3 cazuri de condiții inițiale impuse :

$$(1) \quad X^{(0)} = (0., 0., 0.), \quad EPS = 10^{-3}, \quad ITMAX = 10;$$

$$(2) \quad X^{(0)} = (0., 0., 0.) \quad EPS = 10^{-8}, \quad ITMAX = 10;$$

$$(3) \quad X^{(0)} = (0., 0., 0.), \quad EPS = 10^{-15}, \quad ITMAX = 25.$$

În fig. 6. se dă subprogramul de calcul pentru soluționarea unui sistem neliniar cu 6 ecuații și 6 necunoscute, sistem algebric care descrie regimul permanent de funcționare al unei punți Wheatstone avînd într-un braț un element neliniar de tip termistor. În fig. 7 se prezintă rezultatele obținute pornind inițial de la valori nule pentru curenții prin ramurile punții și impunînd

$$EPS = 10^{-3}, \quad ITMAX = 10.$$

5. **Concluzii.** 1. Se prezintă algoritmul și programul FORTRAN pentru rezolvarea sistemelor de ecuații neliniare prin metoda gradientului.

2. Expresiile finale aduse la forma prezentată în lucrare furnizează un program simplu.

3. Programul testat pe sisteme de dimensiuni mici a dat rezultate foarte bune.

Primită la 20 noiembrie 1978

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Laboratorul de calculatoare

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#### ALGORITHM ET PROGRAMME FORTRAN POUR RÉSOUDRE LES SYSTÈMES ALGÈBRIQUES NONLINÉAIRES

(Résumé)

On présente un algorithme et le programme FORTRAN pour solutionner les systèmes d'équations nonlinéaires à l'aide de la méthode du gradient. Les expressions finales qu'on obtient conduisent à un programme simple. On a testé le programme pour des systèmes de dimension réduite avec des très bons résultats.

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## CONDIȚII OPTIME DE DETERMINARE A DEPLASĂRII ORIZONTALE A CONSTRUCȚIILOR DIN PUNCTUL DE VEDERE AL ERORILOR MEDII PĂTRATICE

DE

GH. NISTOR și ANETA BRAIER

1. Cunoașterea nivelului calitativ al construcțiilor masive se realizează prin completarea calculului de rezistență și stabilitate cu cercetări experimentale de laborator pe modele și cu măsurători și observații pe construcțiile reale, efectuate cu ajutorul aparatelor de măsură și control de înaltă tehnicitate.

Analizarea și interpretarea concomitentă a rezultatelor cercetărilor experimentale de laborator pe modele cu rezultatele calculului de proiectare și cu rezultatele obținute în urma prelucrării măsurătorilor și observațiilor din natură constituie un material prețios pentru realizarea noilor construcții.

Controlul comportării construcțiilor se face prin măsurători topogeodezice ciclice, mai frecvente în timpul execuției și în faza imediat următoare dării în exploatare și din ce în ce mai rare, pe măsură ce deplasările și deformațiile se sting.

Deplasările și deformațiile construcțiilor au în general valori mici, ceea ce face ca măsurarea lor să fie o operație din cele mai dificile. Dar, progresele înregistrate în construcția aparatelor de măsură, ca și îmbunătățirea metodelor de măsurare au permis obținerea de rezultate din ce în ce mai bune.

În lucrarea [2] s-a prezentat o nouă metodă de calcul al deplasărilor orizontale ale construcțiilor, pe baza măsurătorilor efectuate din capetele unei baze de lungime fixă, în funcție directă de variațiile unghiurilor orizontale măsurate în ciclurile de observații actual și inițial. Deoarece la măsurarea deplasărilor și deformațiilor construcțiilor precizia cerută este foarte ridicată, se impune analizarea condițiilor optime de determinare astfel încât erorile să fie minime. Ignorarea acestor condiții duce la situația în care eroarea de determinare este superioară preciziei impuse unor astfel de măsurători sau superioară chiar mărimii deplasărilor sau deformațiilor.

2. Determinarea componentelor deplasării orizontale a punctului de control  $P$  de pe construcția observată se face din capetele  $A$  și  $B$  ale unei baze de lungime constantă (fig. 1) folosind formulele [2]

$$(1) \quad \Delta X = K_2 \Delta \alpha^{\circ\circ} + L_2 \Delta \beta^{\circ\circ},$$

$$(2) \quad \Delta Y = -K_1 \Delta \alpha^{\circ\circ} + L_1 \Delta \beta^{\circ\circ},$$

unde s-au folosit notațiile

$$G = \frac{b}{\rho \sin^2(\alpha + \beta)}, \quad K_1 = -G \sin \beta \cos \beta, \quad K_2 = G \sin^2 \beta, \\ L_1 = G \sin \alpha \cos \alpha, \quad L_2 = G \sin^2 \alpha.$$

Se reamintește că formulele (1) și (2) au caracter de generalitate, mărimile  $K_1$ ,  $L_1$ ,  $K_2$  și  $L_2$  fiind calculate cu elementele măsurate în ciclul inițial și rămânând constante pentru toate ciclurile de observații ulterioare.

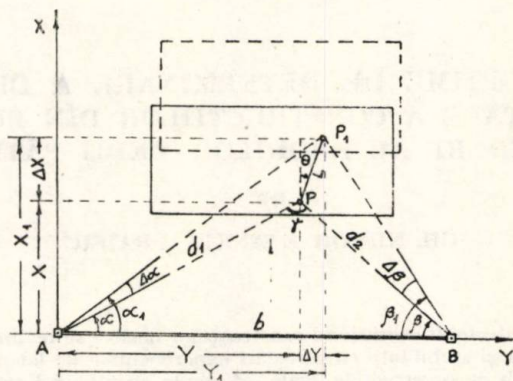


Fig. 1

Mărimea vectorului deplasării orizontale precum și orientarea lui sînt date de relațiile

$$(3) \quad L = \sqrt{\Delta X^2 + \Delta Y^2},$$

$$(4) \quad \theta_L = \arctg \frac{\Delta Y}{\Delta X}.$$

Întrucît mărimile  $K_1$ ,  $L_1$ ,  $K_2$  și  $L_2$  sînt constante, s-a determinat eroarea medie pătratică a vectorului deplasării funcție de mărimile măsurate,  $\Delta\alpha^{\circ\circ}$  și  $\Delta\beta^{\circ\circ}$ , rezultînd

$$(5) \quad m_L = \frac{m_{\Delta\alpha}}{L} \sqrt{\Delta X^2(K_2^2 + L_2^2) + \Delta Y^2(K_1^2 + L_1^2) - 2\Delta X\Delta Y(K_1K_2 - L_1L_2)}.$$

Pentru cazul în care  $\alpha = \beta$  și deci  $d_1 = d_2 = d$ , relația (5) devine

$$(6) \quad m_L = \frac{m_{\Delta\alpha} d \sqrt{2}}{L \sin \gamma} \sqrt{\Delta X^2 \cos^2 \frac{\gamma}{2} + \Delta Y^2 \sin^2 \frac{\gamma}{2}},$$

unde  $\gamma = 200^\circ - 2\alpha$  și  $\gamma/2 = 100^\circ - \alpha$ , cu observația că  $\gamma = [0, 200^\circ]$ .

Din analiza relației (6) rezultă că eroarea medie pătratică a deplasării orizontale a punctului de control este direct proporțională cu distanța de la punctele de stație la punctul de control și cu eroarea medie pătratică de determinare a diferențelor unghiulare  $\Delta\alpha^{\circ\circ}$  și  $\Delta\beta^{\circ\circ}$ , și invers proporțională cu mărimea vectorului deplasării precum și cu sinusul unghiului sub care se intersectează dreptele de determinare.



Pentru evaluarea preciziei rezultatului trebuie analizate mărimile erorilor relative și absolute.

Utilizînd formula (6) se obține pentru eroarea medie pătratică relativă formula

$$(7) \quad \frac{m_L}{d} = \frac{m_{\Delta\alpha} \sqrt{2}}{\rho L \sin \gamma} \sqrt{\Delta X^2 \cos^2 \frac{\gamma}{2} + \Delta Y^2 \sin^2 \frac{\gamma}{2}}.$$

Eroarea medie pătratică absolută se obține prin exprimarea erorii vectorului deplasării orizontale în funcție de lungimea bazei fixe. Pentru aceasta se face în (6) substituția

$$(8) \quad d = \frac{b}{2} \cdot \frac{1}{\sin \frac{\gamma}{2}},$$

obținînd pentru eroarea medie pătratică absolută formula

$$(9) \quad M_L = \frac{m_{\Delta\alpha} b}{\rho L \sqrt{2} \sin \gamma \sin \frac{\gamma}{2}} \sqrt{\Delta X^2 \cos^2 \frac{\gamma}{2} + \Delta Y^2 \sin^2 \frac{\gamma}{2}}.$$

Spre deosebire de cazul determinării unui punct prin intersecție unghiulară înainte, unde optimul se obține prin studierea minimului erorii medii pătratice în poziția punctului determinat [1], în cazul determinării vectorului deplasării orizontale a punctului de control de pe construcția observată acest lucru nu mai este posibil deoarece eroarea medie pătratică depinde, în primul rînd, de poziția vectorului deplasării orizontale față de sistemul rectangular de axe.

În cele ce urmează se analizează trei situații caracteristice, determinate de poziția vectorului deplasării orizontale.

a) *Vectorul deplasării orizontale  $L_1$  paralel cu axa  $X$ , adică*

$$(10) \quad \Delta X = L_1 \quad \text{și} \quad \Delta Y = 0.$$

Substituind (10) în (6) rezultă pentru eroarea medie pătratică relativă expresia

$$(11) \quad \frac{m_{L_1}}{d} = \frac{m_{\Delta\alpha}}{\rho \sqrt{2}} \frac{1}{\sin \frac{\gamma}{2}}.$$

Problema constă în determinarea valorii unghiului dintre dreptele de determinare pentru ca eroarea să fie minimă. Rezultă că

$$(12) \quad \frac{m_{L_1}}{d} = \min. \quad \text{cînd} \quad \sin \frac{\gamma}{2} = \max.,$$

adică  $\gamma/2 = 100^\circ$ .

Deci, minimul erorii relative se obține dacă  $\gamma = 200^\circ$ . Substituind acum (10) în (9) rezultă pentru eroarea medie pătratică absolută expresia

$$(13) \quad M_{L_1} = \frac{m_{\Delta\alpha} b}{\rho \sqrt{2}} \frac{1}{\sin^2 \frac{\gamma}{2}}.$$

Minimul erorii absolute se obține când

$$(14) \quad \sin^2 \frac{\gamma}{2} = \max.,$$

deci pentru  $\gamma = 200^\circ$ .

b) *Vectorul deplasării orizontale  $L_2$  paralel cu axa  $Y$ , respectiv*

$$(15) \quad \Delta X = 0, \quad \Delta Y = L_2.$$

După înlocuirea lui (15) în (16) se obține pentru eroarea relativă expresia

$$(16) \quad \frac{m_{L_2}}{d} = \frac{m_{\Delta\alpha}}{\rho \sqrt{2}} \frac{1}{\cos \frac{\gamma}{2}}.$$

Rezultă că

$$(17) \quad \frac{m_{L_2}}{d} = \min. \quad \text{când} \quad \cos \frac{\gamma}{2} = \max.,$$

adică pentru  $\gamma = 0^\circ$ .

Prin substituirea lui (15) în (9) se obține expresia erorii medii pătratice absolute

$$(18) \quad M_{L_2} = \frac{m_{\Delta\alpha} b}{\rho \sqrt{2}} \frac{1}{\sin \gamma},$$

de unde apare evident că minimul erorii absolute se obține pentru  $\sin \gamma = \max.$ , deci când  $\gamma = 100^\circ$ .

c) *Vectorul deplasării îndreptat pe direcția bisectoarei sistemului rectangular de axe, adică*

$$(19) \quad \Delta X = \Delta Y \quad \text{și} \quad L_3 = \Delta X \sqrt{2}.$$

Se înlocuiește expresia (19) în (6) și se obține formula erorii medii pătratice relative

$$(20) \quad \frac{m_{L_3}}{d} = \frac{m_{\Delta\alpha}}{\rho} \frac{1}{\sin \gamma}.$$

Și în acest caz minimul erorii relative se obține când  $\gamma = 100^\circ$ .

Pentru obținerea formulei de calcul a erorii medii pătratice absolute se înlocuiește expresia (19) în (9) rezultând

$$(21) \quad M_{L_3} = \frac{m_{\Delta\alpha} b}{2\rho} \frac{1}{\sin \gamma \sin \frac{\gamma}{2}}.$$

Minimul erorii absolute se obține atunci când

$$(22) \quad \sin \gamma \sin \frac{\gamma}{2} = \max.,$$



adică pentru  $2\left(\cos \frac{\gamma}{2} - \cos \frac{3\gamma}{2}\right) = \max$ . Anularea derivatei dă ecuația  $\frac{1}{2} \sin \frac{\gamma}{2} \left(3 \cos^2 \frac{\gamma}{2}\right) = 0$ ; are loc maxim în intervalul  $\gamma \in [0, 200^\circ]$  pentru

$$(23) \quad \cos \frac{\gamma}{2} = \frac{1}{\sqrt{3}},$$

respectiv  $\gamma/2 = 60^\circ 82'$  sau  $\gamma = 121^\circ 64'$ .

Pentru funcțiile (11), (16) și (20), care reprezintă erorile relative al vectorului deplasării s-a întocmit tabelul 1. În mod similar tabelul 2 se referă la erorile absolute ale vectorului deplasării, exprimate prin funcțiile (13), (18) și (21).

Tabelul 1

| $\gamma$<br>$m_L$ | $0^\circ$                               | $100^\circ$                              | $200^\circ$                                      |
|-------------------|-----------------------------------------|------------------------------------------|--------------------------------------------------|
| $m_{L_1}$         | $\infty$                                | $\searrow \frac{m_{\Delta\alpha}}{\rho}$ | $\searrow \frac{m_{\Delta\alpha}}{\rho\sqrt{2}}$ |
| $m_{L_2}$         | $\frac{m_{\Delta\alpha}}{\rho\sqrt{2}}$ | $\nearrow \frac{m_{\Delta\alpha}}{\rho}$ | $\nearrow \infty$                                |
| $m_{L_3}$         | $\infty$                                | $\searrow \frac{m_{\Delta\alpha}}{\rho}$ | $\nearrow \infty$                                |

Tabelul 2

| $\gamma$<br>$M_L$ | $0^\circ$                               | $100^\circ$                                        | $200^\circ$                                          |
|-------------------|-----------------------------------------|----------------------------------------------------|------------------------------------------------------|
| $M_{L_1}$         | $\infty$                                | $\searrow \frac{m_{\Delta\alpha} b}{\rho\sqrt{2}}$ | $\searrow \frac{m_{\Delta\alpha} b}{\rho 2\sqrt{2}}$ |
| $M_{L_2}$         | $\frac{m_{\Delta\alpha}}{\rho\sqrt{2}}$ | $\nearrow \frac{m_{\Delta\alpha} b}{\rho\sqrt{2}}$ | $\nearrow \infty$                                    |
| $M_{L_3}$         | $\infty$                                | $\searrow \frac{m_{\Delta\alpha} b}{\rho\sqrt{2}}$ | $\nearrow \infty$                                    |

Analiza datelor din cele două tabele duce la concluzia că atât în cazul erorilor relative, cât și al erorilor absolute, când dreptele de determinare se intersectează sub un unghi drept, eroarea medie pătratică este aceeași, indiferent de poziția vectorului deplasării orizontale în raport cu axele de coordonate.

Pentru valori ale unghiului sub care se intersectează dreptele de determinare, adică pentru  $\gamma \in [0^\circ, 200^\circ]$  s-au construit graficele din fig. 2 și 3, ale mărimilor erorilor.

Din fig. 2 rezultă că graficele erorilor relative  $m_{L_1}/d$  și  $m_{L_2}/d$  ale vectorilor deplasării orizontale dispuși pe direcțiile axelor de coordonate sînt dispuse simetric față de ordonata care trece prin abscisa corespunzătoare lui  $\gamma = 100^\circ$ . Ele prezintă câte un punct de minim pentru  $\gamma = 200^\circ$  și respectiv  $\gamma = 0^\circ$  și au fiecare asimptotă verticală. Cele două erori devin egale când  $\gamma = 100^\circ$ . Curba corespunzătoare erorii relative  $m_{L_3}/d$  a vectorului deplasării orizontale, dispus pe direcția bisectoarei sistemului de axe, are un minim pentru  $\gamma = 100^\circ$  și două asimptote verticale, la  $\gamma = 0^\circ$  și  $\gamma = 200^\circ$ , fiind de asemenea simetrică în raport cu aceeași axă ca și primele două.

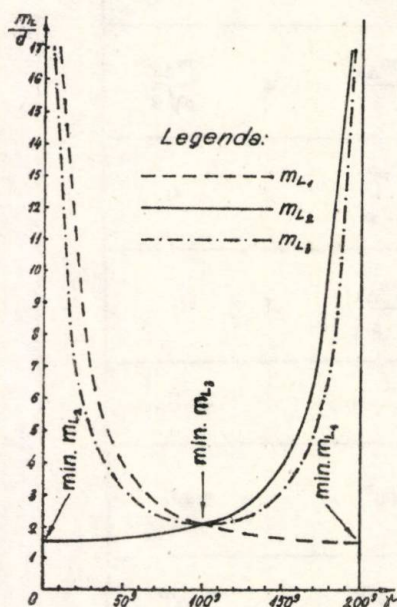


Fig. 2

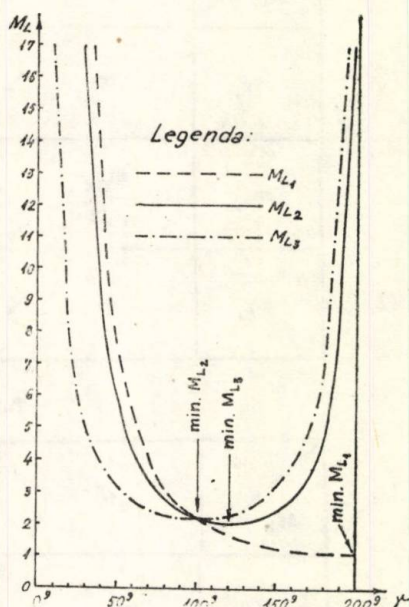


Fig. 3

În fig. 3 sînt reprezentate graficele erorilor absolute. Curba erorii absolute  $M_{L_1}$  corespunzătoare vectorului deplasării orizontale când acesta este dispus paralel cu axa  $X$  are minim pentru  $\gamma = 200^\circ$  și tinde asimptotic spre axa ordonatelor. Curba reprezentativă a erorii  $M_{L_2}$ , care corespunde vectorului deplasării dispus paralel cu axa  $Y$  are un minim pentru  $\gamma = 100^\circ$ , e simetrică în raport cu paralela la axa ordonatelor care trece prin acest punct de minim și prezintă două asimptote verticale. În sfîrșit, curba erorii



absolute  $M_L$ , corespunzătoare vectorului deplasării dispus pe direcția bisectoarei sistemului de axe are un minim pentru  $\gamma \approx 120^\circ$  (mai precis  $\gamma = 121^\circ 64'$ ), prezintă două asimptote verticale dar nu mai are elemente de simetrie. Și în cazul erorilor absolute se constată că pentru  $\gamma = 100^\circ$  curbele au un punct comun, deci aici mărimile acestor erori devin egale.

**3. Concluzii.** 1. Cunoașterea condițiilor optime în care se poate face determinarea deplasărilor construcțiilor supuse acțiunii unor forțe orizontale este de mare importanță în stabilirea a priori a preciziei cu care se va obține mărimea vectorului deplasării orizontale.

2. Atunci când este posibilă cunoașterea anticipată a direcției vectorului deplasării orizontale, baza din capetele căreia se fac măsurătorile asupra punctelor de control de pe construcția observată trebuie astfel aleasă încît unghiul dintre dreptele de determinare să conducă la erori medii pătratice minime. Astfel, la determinarea deplasărilor orizontale ale unui baraj, produse de acțiunea apei din lacul de acumulare, baza se va alege astfel încît unghiul dintre dreptele de determinare să fie apropiat de  $200^\circ$ , valoare pentru care atât erorile relative cît și erorile absolute devin minime.

3. Când nu este posibilă cunoașterea anticipată a direcției vectorului deplasării orizontale se recomandă alegerea bazei astfel încît unghiul dintre dreptele de determinare să fie aproximativ drept. În acest caz erorile medii pătratice relative, ca și cele absolute, au aceleași valori, indiferent de poziția vectorului deplasării față de sistemul rectangular de axe ales.

4. Chiar în cazul respectării condițiilor de determinare a vectorului deplasării, influența hotărîtoare o are precizia de măsurare a diferențelor unghiurilor orizontale, reprezentată de erorile medii pătratice  $m_{\Delta\alpha}$  și  $m_{\Delta\beta}$ , care depind de erorile medii pătratice de măsurare a direcțiilor și unghiurilor în cele două cicluri de observații, inițial și actual.

Primită la 10 mai 1979

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și  
Catedra de geometrie descriptivă  
și desen

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## CONDITIONS OPTIMALES POUR DÉTERMINER LE DÉPLACEMENT HORIZONTAL DES CONSTRUCTIONS, DU POINT DE VUE DES ERREURS QUADRATIQUES MOYENNES

(Résumé)

Dans [2] on a présenté une nouvelle méthode pour le calcul des déplacements horizontaux des constructions à l'aide des mesures faites aux extrémités d'une base de longueur constante. Maintenant on établit les conditions optimales dans lesquelles on doit effectuer les mesures afin que les erreurs en résultent minimales.





# ON CERTAIN DYNAMICAL PROBLEMS IN LINEAR MICROPOLAR ELASTICITY

BY

VITALIE GHEORGHÎĂ

**0. Introduction.** In the last ten years the authors attention was concentrated on the integration of motion equations for some particular boundary and initial value problems of linear micropolar elastodynamics [4] ... [6].

The present paper is concerned with the propagation of elastic waves in an infinite micropolar space with a cylindrical hollow and then with a spherical hollow.

Explicit solutions of the considered problems are given.

**1. Basic Equations.** Equations of linear micropolar elasticity will constitute the starting point of our considerations [1] ... [3].

We shall consider an isotropic homogeneous and centrosymmetric solid. The basic equations of motion written in the vector form are

$$(1.1) \quad \begin{aligned} (\mu + \alpha) \nabla^2 \mathbf{u} + (\lambda + \mu - \alpha) \text{grad div } \mathbf{u} + 2\alpha \text{rot } \boldsymbol{\varphi} + \mathbf{F} &= \rho \ddot{\mathbf{u}}, \\ [(\gamma + \varepsilon) \nabla^2 - 4\alpha] \boldsymbol{\varphi} + (\beta + \gamma - \varepsilon) \text{grad div } \boldsymbol{\varphi} + 2\alpha \text{rot } \mathbf{u} + \mathbf{M} &= J \ddot{\boldsymbol{\varphi}}. \end{aligned}$$

In the above equation  $\mathbf{u}$  denotes the displacement vector,  $\boldsymbol{\varphi}$  is the micro-rotation vector,  $\mathbf{F}$  and  $\mathbf{M}$  stand for the body-force and body-couple respectively,  $\rho$  denotes density,  $J$ —rotational inertia and  $\alpha, \beta, \gamma, \varepsilon, \mu, \lambda$  are material constants. The time derivative of the functions  $\mathbf{u}, \boldsymbol{\varphi}$  is marked by a point.

Using the d'Alembert's and the Klein-Gordon's operators we can rewrite the system (1.1) in the following form :

$$(1.2) \quad \begin{aligned} \square_2 \mathbf{u} + (\lambda + \mu - \varepsilon) \text{grad div } \mathbf{u} + 2\alpha \text{rot } \boldsymbol{\varphi} + \mathbf{F} &= 0, \\ \square_4 \boldsymbol{\varphi} + (\beta + \gamma - \varepsilon) \text{grad div } \boldsymbol{\varphi} + 2\alpha \text{rot } \mathbf{u} + \mathbf{M} &= 0, \end{aligned}$$

where

$$(1.3) \quad \begin{aligned} \square_2 &= (\mu + \alpha) \nabla^2 - \rho \frac{\partial^2}{\partial t^2}, \\ \square_4 &= (\gamma + \varepsilon) \nabla^2 - 4\alpha - J \frac{\partial^2}{\partial t^2}. \end{aligned}$$

The system (1.2) is a coupled system of differential equations in displacements and rotations and in the general case it is very difficult and inconvenient to deal with it. Therefore we shall try to uncouple the equations (1.2) by a method analogous to that used by Lamé in the classical elastokinetics. Following Nowacki [7] we decompose the vectors  $\mathbf{u}$  and  $\varphi$  into the potential and solenoidal parts (this thing is always possible)

$$(1.4)_a \quad \begin{aligned} \mathbf{u} &= \text{grad } \Phi + \text{rot } \Psi, \quad \text{div } \Psi = 0, \\ \varphi &= \text{grad } \Gamma + \text{rot } \mathbf{H}, \quad \text{div } \mathbf{H} = 0. \end{aligned}$$

By the same way, the body forces and moments may be written as

$$(1.4)_b \quad \begin{aligned} \mathbf{F} &= \rho (\text{grad } \Theta + \text{rot } \chi), \quad \text{div } \chi = 0, \\ \mathbf{M} &= J (\text{grad } \sigma + \text{rot } \eta), \quad \text{div } \eta = 0. \end{aligned}$$

Substituting the above relations into (1.2) we get the following, simple wave equations :

$$(1.5) \quad \begin{aligned} \square_1 \Phi + \rho \Theta &= 0, \quad \square_3 \Gamma + J \sigma = 0, \\ \square_2 \Psi + 2\alpha \text{rot } \mathbf{H} + \rho \chi &= 0, \\ \square_4 \mathbf{H} + 2\alpha \text{rot } \Psi + J \eta &= 0, \end{aligned}$$

where we have introduced the operators

$$(1.6) \quad \begin{aligned} \square_1 &= (\lambda + 2\mu) \nabla^2 - \rho \frac{\partial^2}{\partial t^2}, \\ \square_3 &= (\beta + 2\gamma) \nabla^2 - 4\alpha - J \frac{\partial^2}{\partial t^2}. \end{aligned}$$

The equations (1.5) have the following signification : the first represents the equation of longitudinal wave, the second is the equation of longitudinal, microrotational wave, the third and the fourth equations describe the propagation of the displacement shear wave and the microrotational shear wave, respectively.

In the particular case of absence of the body and moment-forces the system (1.5) becomes

$$(1.7) \quad \begin{aligned} \square_1 \Phi &= 0, \quad \square_3 \Gamma = 0, \\ \square_2 \Psi + 2\alpha \text{rot } \mathbf{H} &= 0, \\ \square_4 \mathbf{H} + 2\alpha \text{rot } \Psi &= 0. \end{aligned}$$

The last two equations of (1.7) after the elimination of  $\Psi$  and  $\mathbf{H}$  may be reduced to the following independent equations :

$$(1.8) \quad \begin{aligned} (\square_2 \square_4 + 4\alpha^2 \nabla^2) \Psi &= 0, \\ (\square_2 \square_4 + 4\alpha^2 \nabla^2) \mathbf{H} &= 0. \end{aligned}$$

These equations have been investigated by J. Ignaczak [8].



In what follows it will be more convenient to write (1.2) in the framework of cylindrical coordinates  $(r, \theta, z)$ . We need here only the equations (1.2) corresponding to the particular case when the vectors  $\mathbf{u}$  and  $\varphi$  are independent of the angle  $\theta$ , and  $u_\theta = \varphi_r = \varphi_z = 0$ ,  $\mathbf{F} = \mathbf{M} = 0$ . From the system of equations (1.2) remain only three equations which in cylindrical coordinates take the form

$$\begin{aligned}
 & (\mu + \alpha) \left( \nabla^2 u_r - \frac{u_r}{r^2} \right) + (\lambda + \mu - \alpha) \frac{\partial e}{\partial r} - 2\alpha \frac{\partial \varphi_\theta}{\partial z} = \rho \ddot{u}_r, \\
 (1.9) \quad & (\mu + \alpha) \nabla^2 u_z + (\lambda + \mu - \alpha) \frac{\partial e}{\partial z} + 2\alpha \frac{1}{r} \frac{\partial}{\partial r} (r \varphi_\theta) = \rho \ddot{u}_z, \\
 & (\gamma + \varepsilon) \left( \nabla^2 \varphi_\theta - \frac{\varphi_\theta}{r^2} \right) - 4\alpha \varphi_\theta + 2\alpha \left( \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) = J \ddot{\varphi}_\theta,
 \end{aligned}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \quad e = \frac{1}{r} \frac{\partial}{\partial r} (r u_r) + \frac{\partial u_z}{\partial z}, \quad r = (x_1^2 + x_2^2)^{1/2}.$$

We shall now write the equations (1.2) in the spherical coordinates system  $(R, \theta, \eta)$ . Assuming  $u_\eta = \varphi_R = \varphi_\theta = 0$ ,  $\mathbf{F} = \mathbf{M} = 0$  and  $u_R, u_\theta, \varphi_\eta$  are independent of  $\theta$ , we obtain the following equations describing the displacements  $\mathbf{u} = (u_R, u_\theta, 0)$  and the rotations  $\varphi = (0, 0, \varphi_\eta)$ :

$$\begin{aligned}
 & (\mu + \alpha) \left\{ \nabla^2 u_R - \frac{2}{R^2} \left[ u_R + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (u_\theta \sin \theta) \right] \right\} + (\lambda + \mu - \alpha) \frac{\partial e}{\partial R} + \\
 & \quad + \frac{2\alpha}{R \sin \theta} \frac{\partial}{\partial \theta} (\varphi_\eta \sin \theta) = \rho \ddot{u}_R, \\
 (1.10) \quad & (\mu + \alpha) \left\{ \nabla^2 u_\theta - \frac{2}{R^2} \left[ \frac{\partial u_R}{\partial \theta} - \frac{u_\theta}{2 \sin^2 \theta} \right] \right\} + (\lambda + \mu - \alpha) \frac{1}{R} \frac{\partial e}{\partial R} - \\
 & \quad - \frac{2\alpha}{R} \frac{\partial}{\partial R} (R \varphi_\eta) = \rho \ddot{u}_\theta, \\
 & (\gamma + \varepsilon) \left\{ \nabla^2 \varphi_\eta - \frac{\varphi_\eta}{R^2 \sin^2 \theta} \right\} - 4\alpha \varphi_\eta - 2\alpha \left( \frac{1}{R} \frac{\partial u_R}{\partial \theta} - \frac{1}{R} \frac{\partial}{\partial R} (R u_\theta) \right) = J \ddot{\varphi}_\eta.
 \end{aligned}$$

Here the following notations have been used:

$$\begin{aligned}
 \nabla^2 &= \frac{\partial^2}{\partial R^2} + \frac{2}{R^2} \frac{\partial}{\partial R} + \frac{1}{R^2 \sin^2 \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right), \\
 e &= \frac{1}{R^2} \frac{\partial}{\partial R} (R^2 u_R) + \frac{1}{R \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta u_\theta), \quad R = (x_1^2 + x_2^2 + x_3^2)^{1/2}.
 \end{aligned}$$

## 2. Propagation of Elastic Waves in an Infinite Space with a Hole.

Let us consider an infinite space with a cylindrical hole of circular section made of isotropic, homogeneous, centrosymmetric material. We assume that on the surface of the cylinder  $r = a$ , acts a pressure  $p(t)$  varying harmonical with respect to time ( $p = p_0 e^{i\omega t}$ ) and is uniformly distributed on the boundary surface. In this case only the cylindrical longitudinal waves are generated. We have in fact an axisymmetric problem and therefore we use the equations (1.9) in which the displacements  $u_r, u_z$  and the rotation  $\varphi_\theta$  may be expressed by means of potentials  $\Phi = \Phi(r)$ ,  $\Gamma = \Gamma(r)$ . Taking into account (1.4)<sub>a</sub> the system of equations (1.9) reduces to a single differential equation

$$(2.1) \quad \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} - \frac{1}{c_1^2} \ddot{\Phi} = 0, \quad c_1^2 = \frac{\lambda + 2\mu}{\rho}.$$

We have to integrate the equation (2.1) under the boundary conditions

$$(2.2) \quad t_{rr}|_{r=a} = -p_0 e^{i\omega t}, \quad t_{rr} = 0 \quad \text{for} \quad r \rightarrow \infty,$$

where  $t_{ij}$  denotes the stress tensor components and  $a$  is the radius of the cylinder. Obviously, the potential  $\Phi$  will have the same form as the perturbation has i.e.  $\Phi(r, t) = \Phi^*(r) e^{i\omega t}$ .

Substituting  $\Phi$  into (2.1) we obtain

$$(2.3) \quad \frac{d^2 \Phi^*}{dr^2} + \frac{1}{r} \frac{d\Phi^*}{dr} + \frac{\omega^2}{c_1^2} \Phi^* = 0.$$

The solution of (2.3) is of the form

$$(2.4) \quad \Phi^*(r) = AK_0\left(\frac{i\omega r}{c_1}\right) + BI_0\left(\frac{i\omega r}{c_1}\right),$$

where  $K_0$  and  $I_0$  are the modified Bessel functions of order zero.

Taking into account the behaviour of the Bessel functions at infinity and the second condition (2.2) it results

$$(2.5) \quad \Phi^*(r) = AK_0\left(\frac{i\omega r}{c_1}\right).$$

Finally, we get

$$(2.6) \quad \Phi(r) = AK_0\left(\frac{i\omega r}{c_1}\right) e^{i\omega t}.$$

Calculating the stress  $t_{rr}$  according to the formula

$$t_{rr} = 2\mu \frac{\partial u_r}{\partial r} + \lambda \left( \frac{\partial u_r}{\partial r} + \frac{u_r}{r} \right) = 2\mu \frac{\partial^2 \Phi}{\partial r^2} + \frac{\lambda}{c_1^2} \frac{\partial^2 \Phi}{\partial t^2}$$

and imposing to be satisfied the boundary condition (2.2), we find



$$(2.7) \quad A = \frac{p_0}{\frac{\omega^2}{c_1^2} \lambda K_0 \left( \frac{i\omega a}{c_1} \right) - 2\mu \left| \frac{d^2}{dr^2} K_0 \left( \frac{i\omega r}{c_1} \right) \right|_{r=a}}.$$

The problem of determining the stress and strain states corresponding to the harmonic potential  $\Phi$  is now a simple task.

Let us consider an analogous problem but at this time the infinite space has a spherical hole with the radius  $R_0$ . The pressure  $p = p_0 e^{i\omega t}$  acts on the surface of sphere  $R = R_0$ . In space appear only longitudinal and dilatation waves. The posed problem is a case of symmetry with respect to the point and that is why we use the equations (1.10) where  $u_\theta = \varphi_\eta = 0$ . Thus, what remains from (1.10) is the first equation which takes the form

$$(2.8) \quad (\lambda + 2\mu) \left( \nabla_R^2 - \frac{2}{R^2} \right) u_R - \rho \frac{\partial^2 u_R}{\partial t^2} = 0,$$

$$\nabla_R^2 = \frac{\partial^2}{\partial R^2} + \frac{2}{R} \frac{\partial}{\partial R}.$$

If we express the displacement  $u_R$  by means of the potential  $\Phi = \Phi(R)$  as  $u_R = \frac{\partial \Phi}{\partial R}$ , then the equation (2.8) becomes

$$(2.9) \quad \frac{\partial^2 \Phi}{\partial R^2} + \frac{2}{R} \frac{\partial \Phi}{\partial R} - \frac{1}{c_1^2} \frac{\partial^2 \Phi}{\partial t^2} = 0.$$

Putting  $\Phi(R, t) = \Phi^*(R) e^{i\omega t}$  into (2.9) we arrive at the equation

$$(2.10) \quad \frac{d^2 \Phi^*}{dR^2} + \frac{2}{R} \frac{d\Phi^*}{dR} + \frac{\omega^2}{c_1^2} \Phi^* = 0.$$

The following boundary conditions are to be satisfied :

$$(2.11) \quad t_{RR} = -p_0 e^{i\omega t}, \quad t_{RR} = 0 \quad \text{for} \quad R \rightarrow \infty.$$

The second condition (2.11) implies  $\Phi^* = 0$  for  $R \rightarrow \infty$ . It is well-known that a solution of (2.10) which is bounded at infinity has the form

$$(2.12) \quad \Phi^*(R) = \frac{A_1}{R} e^{-\frac{i\omega R}{c_1}}.$$

Finally, it results

$$(2.13) \quad \Phi(R, t) = \frac{A_1}{R} e^{i\omega \left( t - \frac{R}{c_1} \right)}.$$

From the first boundary condition (2.11) by using the formula

$$t_{RR} = 2\mu \frac{\partial u_R}{\partial R} + \lambda \left( \frac{\partial u_R}{\partial R} + \frac{2}{R} u_R \right) = 2\mu \frac{\partial^2 \Phi}{\partial R^2} + \lambda \left( \frac{\partial^2 \Phi}{\partial R^2} + \frac{2}{R} \frac{\partial \Phi}{\partial R} \right),$$

we obtain

$$(2.14) \quad A_1 = \frac{p_0 R_0^2 e^{\frac{i\omega R_0}{c_1}}}{\rho\omega^2 R_0 - 4\mu \left( \frac{1}{R_0} + \frac{i\omega}{c_1} \right)}.$$

The potential  $\Phi(R, t)$  — completely determined — allows to calculate the stress  $t_{RR}$  by the above formula and also the stresses  $t_{\theta\theta}$ ,  $t_{\eta\eta}$  by

$$(2.15) \quad t_{\theta\theta} = t_{\eta\eta} = 2\mu \frac{\partial \Phi}{\partial R} + \lambda \left( \frac{\partial^2 \Phi}{\partial R^2} + \frac{2}{R} \frac{\partial \Phi}{\partial R} \right).$$

The question of finding the potential  $\Phi$  in the case of arbitrary varying of pressure  $p$  with respect to time is — generally speaking — too complicated.

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#### DESPRE UNELE PROBLEME DINAMICE ÎN ELASTICITATEA MICROPOLARĂ LINIARĂ

(Rezumat)

Se studiază propagarea undelor elastice într-un solid micropolar, izotrop, omogen și centrosimetric care ocupă un spațiu infinit cu un gol de formă cilindrică sau sferică.

Se dau soluții explicite ale problemelor considerate în cazul în care forța perturbatoare este o presiune variind armonic în timp.



# THE PRINCIPLE OF EQUIVALENCE, THE GRAVITATIONAL COMPASS AND THE MEASUREMENT OF GRADIENTS IN THE GRAVITATIONAL FIELD

BY

CORNELIU D. CIUBOTARIU

**1. Introduction.** The constancy of the ratio of inertial mass to gravitational mass, which we will denote here by  $\alpha$ , had been investigated since the time of Galileo, who observed that, in the earth's gravitational field, all bodies fall with the same acceleration. In 1830 Bessel has been considered the influence of  $\alpha$  on the period of pendulum. For a simple pendulum one has from Newton's laws  $T = 2\pi (l/g)^{1/2}$ , where  $l$  is the length of the pendulum, but if inertial mass and gravitational mass were not equal one would have  $T = 2\pi (m_i l/m_g g)^{1/2} = 2\pi (\alpha l/g)^{1/2}$ , where  $m_i$ ,  $m_g$  are the inertial and gravitational masses respectively. Clearly as we change  $m_g$  unless  $m_i$  change proportionately, the period will also change. By this method Bessel verified the principle of equivalence to a few parts in  $10^5$ . The next significant improvement in accuracy came with the work of Eötvös who demonstrated experimentally the equivalence of inertial mass and gravitational mass to about one part in  $10^8$ .

In the present Note we will show that the principle of equivalence may be verified with a Cavendish balance, with different material at opposite ends of the beam, without suspension fiber. If inertial mass and gravitational mass are not equal the new balance is the most simple gravitational compass. Finally, a method for the measurement of gradients in gravitational field is indicated.

**2. The Gravitational Compass.** For easy reference we give the strategy of Eötvös experiment (see, for example [1]). In Eötvös' instrument, consisting of a Cavendish balance with different material at opposite ends of the beam, the centrifugal forces on the two masses (1 and 2) due to the earth's rotation ( $\omega \cong 0,73 \times 10^{-4} \text{ rad s}^{-1}$ ) are balanced against the earth's gravitational field (Fig. 1). If the ratio  $\alpha$  should differ from one mass to the other, there would be a torque tending to twist the torsion balance:

$$(1) \quad \mathbf{f} \cdot \mathbf{M} = 2R\omega^2 m_{i1} m_{i2} (\alpha_2 - \alpha_1) (\alpha_1 m_{i1} + \alpha_2 m_{i2})^{-1} \mathbf{d} \cdot (\mathbf{C} \times \mathbf{G}) \cos \varphi,$$

where  $R$  is the earth's ray,  $\varphi$  is the latitude,  $m_i$  is the inertial mass,  $\mathbf{d}$  is the radius vector from the torsion fiber axis to a mass, and the  $\mathbf{G}$  and  $\mathbf{C}$  are the unitary vectors of the gravitational and centrifugal forces respectively.  $\mathbf{f}$  is

the direction of the suspension fiber, and  $M$  is the total mechanical moment of the gravitational and centrifugal forces. If  $\alpha_1 \neq \alpha_2$ , the magnitude of  $f \cdot M$  depends of the direction of  $d$  relative to meridian plane  $C \times G$ .  $f \cdot M$  is maximum when  $d$  is in west-east direction and it cancels for a torsion balance beam in a north-south direction.

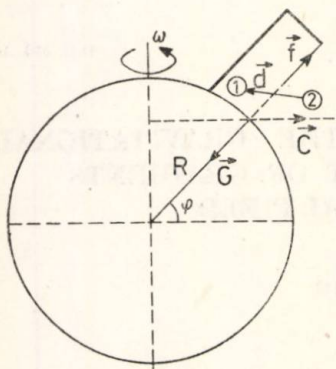


Fig. 1

Hence, there is a permanent gravitational-inertial torque (unless north-south direction of the beam) which acts on a rod with different materials at opposite ends and which tends to orientate the rod on the geographical north direction. If the rod is suspended from a fiber the end of which is fixed to a support point without friction, with a superfluid or a magnetic suspension, it becomes a gravitational compass which works only if  $\alpha_1 \neq \alpha_2$ .

At this moment we remark that it is not necessary a torsion suspension fiber. The balance beam may verify simply the equivalence principle if it is suspended in a superfluid medium. At low temperature the various disturbances would be minimized and the precision of the experiment will increase. The mass  $m_{i1}$  and  $m_{i2}$  may be of great value and thus one may improve too the accuracy of the principle of equivalence. One may defines, then, a gravitational declination and a gravitational inclination.

It is interesting to analyse the Einstein's famous falling-elevator experiment in the case of a freely falling rod with different materials at opposite ends. If  $\alpha_1 \neq \alpha_2$  the equation of motion is dependent of the mass of the body and thus the two materials do not fall in the same way, the rod will suffer some deviation from his initial „horizontal„ direction. An interesting experiment will be to place such a rod in an artificial satellite. The effect of  $\alpha_1 \neq \alpha_2$  will be increased by the large force anomaly ( $f \cdot M \sim \omega^2$ ) combined with the long observation time.

Pirani [2] has suggested that the physical components of the Riemann tensor can be determined by experiments consisting of throwing up clouds of test particles and measuring the relative accelerations between them. We suggest that if the clouds are formed of different test particles and  $\alpha_1 \neq \alpha_2$  the relative accelerations will exist even in an uniform gravitational field and this is a possibility to verify the principle of equivalence.

**3. The Measurement of Gradients in the Gravitational Field.** Let us consider a torsion balance of Eötvös type [3] (see Fig. 2) which consists of a light horizontal beam suspended by a thin wire. Attached to the beam are two weight, one suspended below the other. Any small rotation of the beam can be observed by an optical lever detection system.

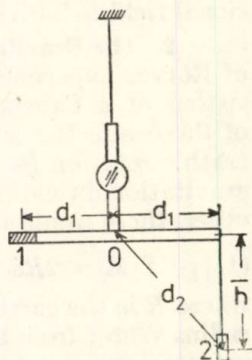


Fig. 2



In order to calculate the resulting torque on the balance due to the gravitational field gradients we make the following obvious notations:

$$(2) \quad \mathbf{G}_1 = m_{g1} g_1 \mathbf{G}, \quad \mathbf{G}_2 = m_{g2} g_2 \mathbf{G},$$

(gravitational forces;  $\mathbf{G}$  is an unitary vector),

$$(3) \quad \mathbf{C}_1 = m_{i1} \omega^2 R_1 \cos \varphi \mathbf{C}, \quad \mathbf{C}_2 = m_{i2} \omega^2 R_2 \cos \varphi \mathbf{C},$$

(centrifugal forces;  $\mathbf{C}$  is an unitary vector).

The direction of the suspension fiber is that of the resulting force which has the unitary vector

$$(4) \quad \mathbf{f} = \frac{(m_{g1} g_1 + m_{g2} g_2) \mathbf{G} + (m_{i1} R_1 + m_{i2} R_2) \omega^2 \cos \varphi \mathbf{C}}{|\mathbf{G}_1 + \mathbf{G}_2 + \mathbf{C}_1 + \mathbf{C}_2|}.$$

The component of the resulting torque along the suspension fiber is given by

$$(5) \quad \begin{aligned} \mathbf{f} \cdot \mathbf{M} = & \frac{2 (m_{i1} m_{g2} g_2 R_1 - m_{i2} m_{g1} g_1 R_2) \omega^2 \cos \varphi \mathbf{d}_1 \cdot (\mathbf{C} \times \mathbf{G})}{|\mathbf{G}_1 + \mathbf{G}_2 + \mathbf{C}_1 + \mathbf{C}_2|} + \\ & + \frac{(m_{g1} m_{i2} g_1 R_2 - m_{g2} m_{i1} g_2 R_1) \omega^2 \cos \varphi \mathbf{h} \cdot (\mathbf{C} \times \mathbf{G})}{|\mathbf{G}_1 + \mathbf{G}_2 + \mathbf{C}_1 + \mathbf{C}_2|}. \end{aligned}$$

Next, for the vector  $\mathbf{h}$  we have

$$(6) \quad \mathbf{h} = h \frac{m_{g2} g_2 \mathbf{G} + m_{i2} \omega^2 R_2 \cos \varphi \mathbf{C}}{|\mathbf{G}_2 + \mathbf{C}_2|}$$

and therefore

$$(7) \quad \mathbf{h} \cdot (\mathbf{C} \times \mathbf{G}) = 0.$$

We have found that the considered balance is insensitive to vertical gravitational gradients. A small sensitivity may appear only if the torsion beam is very long, that is, if

$$(8) \quad \frac{\mathbf{G}_1}{m_{g1} g_1} = \frac{\mathbf{G}_2}{m_{g2} g_2}.$$

However, this apparatus may be used to measure the relative variations of the gravitational acceleration if we choose simply (equivalence principle)

$$(9) \quad m_{i1} = m_{i2} = m_{g1} = m_{g2}, \quad R_1 = R_2,$$

and thus

$$\mathbf{f} \cdot \mathbf{M} \cong \text{const.} \frac{g_2 - g_1}{g_2 + g_1}.$$

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PRINCIPIUL ECHIVALENȚEI, BUSOLA GRAVITAȚIONALĂ ȘI MĂSURAREA  
GRADIENTILOR ÎN CÂMPUL GRAVITAȚIONAL

(Rezumat)

Se arată că principiul echivalenței poate fi verificat cu o balanță Cavendish fără fir de suspensie și având mase de probă diferite la extremitățile brațului orizontal. Dacă masa inertă și masa gravitațională nu sînt egale, noua balanță reprezintă cea mai simplă busolă gravitațională. Se discută o metodă pentru măsurarea gradientilor în câmpul gravitațional.



## REMARKS ON THE RELATIVISTIC INVARIANTIVE MAGNETOFLUID DYNAMICS

BY

I. MERCHES and M. AGOP

1. Introduction. Let us have the following differential expression

$$(1) \quad \omega(d) = X_i dx^i, \quad i = 1, 2, \dots, h,$$

where the functions  $X_i = X_i(x^k)$  are supposed to be regular in a certain domain in  $x^1, x^2, \dots, x^h$  space. The expression (1) is usually called a Pfaff form. Let us also have the Pfaff form

$$(2) \quad \omega(\delta) = X_i \delta x^i,$$

where the variations  $d$  and  $\delta$  are independent. The expression

$$(3) \quad \Delta\omega = d\omega(\delta) - \delta\omega(d) = T_{ij} \delta x^i dx^j,$$

where  $T_{ij}$  is a second rank covariant antisymmetric tensor, is usually called the bilinear covariant of the form  $\omega$  or its exterior derivative. If  $\omega$  is an exact differential, then  $\Delta\omega = 0$ .

Using the fundamental ideas of the invariantive mechanics [1], [2], the vanishing  $\Delta\omega$  for any  $\delta$  will lead to the desired equations of motion.

On the other hand, it can be shown [3] that the relativistically covariant equation of motion of a charged particle in an external electromagnetic field can be written as

$$(4) \quad \frac{d}{d\tau} (m_0 u_i) + m_0 R_{ik} u^k = 0,$$

where

$$(5) \quad R_{ik} = R_{k,i} - R_{i,k}, \quad R_i = \rho_0^{-1} F_{ik} P_0^k.$$

Here  $\rho_0$  is the rest mass density,  $F_{ik}$  is the electromagnetic field tensor, and  $P_0^k$  is the polarization four-vector, taken in the proper coordinate system. It also can be shown [3] that the equation of motion of a charged fluid takes the form

$$(6) \quad u^k \frac{\partial}{\partial x^k} (c^{-2} w u_i) = -\rho_0^{-1} \frac{\partial p}{\partial x^i} - R_{ik} u^k,$$

where  $w$  is the specific enthalpy. These two equations have been obtained by using a variational formulation.

The purpose of the present paper is to obtain the equations of motion (4) and (6), written both in Galilean and general coordinate systems. This will be done by making allowance for the methods of invariantive mechanics.

**2. The Equation of Motion of a Charged Particle.** a) *Galilean coordinates.* We postulate that the motion is given by the invariant differential form

$$(7) \quad \omega(d) = p_i dx^i,$$

where  $p_i$  is the energy-momentum four-vector

$$(8) \quad p_i = m_0 (u_i - R_i).$$

The bilinear covariant of (7) is

$$(9) \quad \begin{aligned} \Delta\omega &= m_0 (du_i^t \delta x^t - \delta u_i dx^t + \delta R_i dx^t - dR_i \delta x^t) = \\ &= m_0 du_i \delta x^t - m_0 \delta u_i dx^t + m_0 R_{ik} \delta x^t dx^k, \end{aligned}$$

with  $R_i$  and  $R_{ik}$  given by (5). Dividing now (9) by the proper time  $d\tau$ , we get

$$(10) \quad \frac{\Delta\omega}{d\tau} = m_0 \left( \frac{du_i}{d\tau} + R_{ik} u^k \right) \delta x^t - m_0 u_i \delta u_i.$$

Since  $u_i u^i = -c^2$ ,  $u_i \delta u^i = 0$ , and the condition of vanishing  $\Delta\omega/d\tau$  leads to the desired equation of motion (4).

b) *General coordinates.* We shall take now  $\omega(d) = p^i dx_i$ ,  $\omega(\delta) = p^i \delta x_i$ , so that

$$(11) \quad \Delta\omega = D(p^i \delta x_i) - \bar{D}(p^i dx_i),$$

where  $D$  denote the absolute differential, the symbols  $D$  and  $\bar{D}$  being independent of each other. We may write

$$(12) \quad \Delta\omega = Dp^i \delta x_i - \bar{D}p^i dx_i + p^i D\delta x_i - p^i \bar{D}dx_i.$$

But

$$p^i (D\delta x_i - \bar{D}dx_i) = \Gamma_{ij}^k p^i (dx_k \delta x^j - \delta x_k dx^j)$$

and Eq. (12) reads

$$(13) \quad \Delta\omega = dp^i \delta x_i - \delta p^i dx_i.$$

In view of (8) and (13) we may write

$$(14) \quad \begin{aligned} \Delta\omega &= m_0 \left\{ (du^i + \Gamma_{jk}^i u^j dx^k) \delta x_i - (\delta u^i + \Gamma_{jk}^i u^j \delta x^k) dx_i + \right. \\ &+ \left( \frac{\partial R^i}{\partial x^k} + \Gamma_{jk}^i R^j \right) \delta x^k dx_i - \left( \frac{\partial R^i}{\partial x^k} + \Gamma_{jk}^i R^j \right) dx^k \delta x_i \Big\} = \\ &= m_0 \{ (du^i + \Gamma_{jk}^i u^j dx^k) \delta x_i - (\delta u^i + \Gamma_{jk}^i u^j \delta x^k) dx_i + \\ &+ (\nabla_i R_k - \nabla_k R_i) dx^k \delta x^i \}, \end{aligned}$$

where the symbol  $\nabla_i$  stands for the covariant derivative. But  $\nabla_i R_k - \nabla_k R_i = \partial_i R_k - \partial_k R_i$  and the condition  $\Delta\omega/d\tau \equiv 0$  finally leads to



$$(15) \quad m_0 \left( \frac{du^i}{d\tau} + \Gamma_{jk}^i u^j u^k \right) + m_0 R^{ik} u_k = 0,$$

which is the equation of motion of a charged particle written in general coordinates, in terms of the antisymmetric tensor  $R_{ik}$ .

**3. The Equation of Motion of a Charged Fluid.** a) *Galilean coordinates.* We want to write the equation of motion of an ideal, charged fluid, undergoing isentropic motion in an external electromagnetic field. Following the same procedure, we shall take as the generalized energy-momentum four-vector

$$(16) \quad p_i = \rho_0 w c^{-2} u_i - \rho_0 R_i, \quad$$

where the specific enthalpy  $w$  is related to the specific internal energy  $\varepsilon$  through the relation

$$(17) \quad w = \varepsilon_0 + \frac{p}{\rho_0}.$$

The quantities  $w$ ,  $\varepsilon_0$ , and  $\rho_0$  are taken in the rest frame, while the form of  $p_i$  is suggested by the variational formulation of the problem (see [3]). We have then

$$\Delta\omega = \rho_0 \{ c^{-2} u_i (dw \delta x^i - \delta w dx^i) + c^{-2} w (du_i \delta x^i - \delta u_i dx^i) + (\delta R_i dx^i - dR_i \delta x^i) \},$$

or, after some manipulation

$$(18) \quad \Delta\omega = \rho_0 \{ c^{-2} d(wu_i) \delta x^i + R_{ik} dx^k \delta x^i - c^{-2} \delta w u_i dx^i - w c^{-2} \delta u_i dx^i \}.$$

Since the motion is isentropic,

$$(19) \quad T dS = d\varepsilon_0 + p d\left(\frac{1}{\rho_0}\right) = dw - \frac{dp}{\rho_0} = 0$$

and the condition  $\Delta\omega/d\tau \equiv 0$  gives

$$(20) \quad \left\{ u^k \frac{\partial}{\partial x^k} (c^{-2} w u_i) + \rho_0^{-1} \frac{\partial p}{\partial x^i} + R_{ik} u^k \right\} \delta x^i - w c^{-2} u^i \delta u_i = 0.$$

Using once again the condition  $u_i u^i = -c^2$ , Eq. (20) finally leads to the desired Eq. (6).

b) *General coordinates.* Introducing (16) in (11) and by virtue of (13), one obtains

$$(21) \quad \Delta\omega = \rho_0 \{ c^{-2} d(wu_i) \delta x^i - c^{-2} \delta(wu_i) dx^i + \delta R_i dx^i - dR_i \delta x^i \}.$$

Observing that

$$\begin{aligned} \nabla_j R_i \delta x^j dx^i - \nabla_j R_i dx^j \delta x^i &= R_{ij} dx^j \delta x^i, \\ c^{-2} \rho_0 d(wu_i) \delta x^i &= \nabla_j (c^{-2} \rho_0 w u_i dx^j) \delta x^i, \\ c^{-2} \rho_0 \delta(wu_i) dx^i &= -\nabla_j p g_i^j dx^i, \end{aligned}$$

the already known procedure leads to the equation

$$(22) \quad \nabla_j (c^{-2} \rho_0 w u_i u^j + p g_i^j) + R_{ij} u^j = 0,$$

which is the equation of motion of the considered model of fluid, written in general coordinates.

**4. Conclusions.** i) The method provided by the invariantive mechanics is a useful tool in magnetofluid dynamics.

ii) The above considerations prove once again the usefulness of the representation of the electromagnetic field in terms of  $R_i$ . The use of  $R_i$  instead of  $A_i$  seems to be more adequate when the physical problem implies a flux of charged particles (or a charged fluid).

iii) The quantity  $c^{-2} \rho_0 w u_i u^k + p g_i^k$  is the energy-momentum tensor of the fluid. If the electromagnetic field is not present, then Eqs. (6) and (22) turn into the well-known equations of hydrodynamics.

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#### CONSIDERAȚII ASUPRA MAGNETOFLUIDODINAMICII INVARIANTIVE RELATIVISTE

(Rezumat)

Plecînd de la ideile de bază ale mecanicii invariantive și utilizînd noțiunea de cuadri-vector antipotențial generalizat al cîmpului electromagnetic, se obțin ecuațiile de mișcare ale unei particule încărcate și ale unui fluid încărcat, atât în coordonate galileene cît și în coordonate generale.



# ON THE CHOICE OF THE PRE-EXCITATION OF THE LIQUID CRYSTAL DEVICES USED IN TRANSDUCERS

BY

EMIL LUCA, HORIA-NICOLAI TEODORESCU, EMIL SOFRON and I. BACIU

**1. Generalities.** The liquid crystals (LC) are very sensitive to electromagnetic, thermic and mechanic energy and to chemical agents [1], [2], [3], [4]. The nematic liquid crystals, used in devices based on the dynamic scattering effect (DS-N-LCD), change their electrical and/or optical properties when they are submitted to electromagnetic fields (IR, UV etc.), to thermal influences or to mechanical forces, pressures, vibrations and displacements. Theoretical and experimental facts indicate that the regime of the DS-N-LCD is important in obtaining the best sensitivity of the device to these parameters. In what follows are discussed some problems connected to the suitable use of the N-LCDs in transducers.

**2. The Electrical Characteristics of the DS-N-LCDs.** The devices with N-LC represent the input circuit of the transducers and from this point of view their impedance and their current versus voltage characteristics are important for the general characteristics of the transducers.

In the first approximation, a NLCD may be equivalated to the circuit indicated in Fig. 1. The resistance  $R$  is the series resistance of the NLC layer, the capacitance  $C$  is given by the electrodes and the inductance  $L$  is given by the connection wires and may be neglected in the low frequency domain used for LCDs. In this paper are described the results obtained with two LCDs, denoted LCD 1 and LCD 2, with MBBA, differently impurified. For these devices, the values of the elements of the equivalent circuits are: LCD 1:  $R = 21 \text{ K}\Omega$ ; LCD 2:  $R = 180 \text{ K}\Omega$ ;  $C = 1,9 \text{ nF}$ . The experimental characteristics  $Z = Z(f)$ , where  $Z$  is the impedance of the device and  $f$  is the frequency of the applied signal, confirm the frequency behaviour predicted by the circuit shown in Fig. 1. In Fig. 2 the characteristics  $Z = Z(f)$  for DCL 2 is plotted.

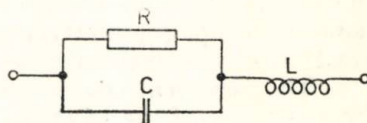


Fig. 1—Equivalent circuit of the N-LCD.

The resistance  $R$  of the equivalent scheme is importantly influenced by the temperature, the capacitance  $C$  remaining constant, as indicated

by the curves current/voltage for various frequencies and temperatures (see Fig. 3). The temperature coefficient is  $\alpha_{20^{\circ}\text{C}, 20\text{Hz}} \approx 3\%/^{\circ}\text{C}$  for DCL 1 and  $\alpha_{20\text{Hz}, 30^{\circ}\text{C}} \approx 2,5\%/^{\circ}\text{C}$  for DCL 2.

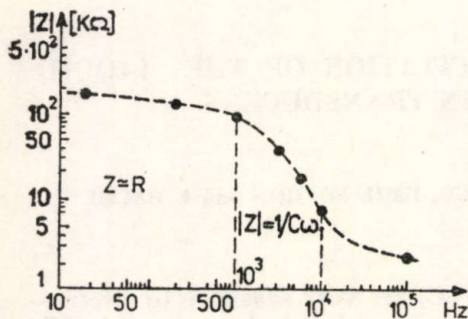


Fig. 2 — Experimental frequency—impedance characteristic of the LCD 2.

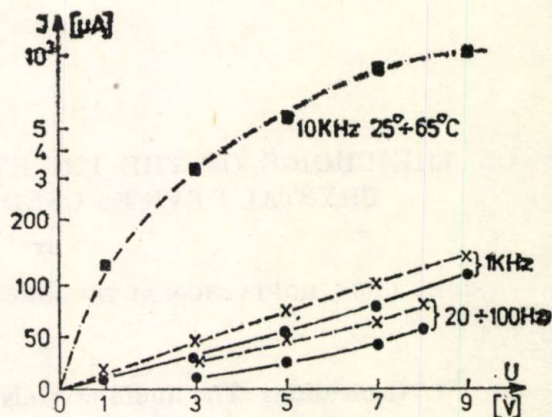


Fig. 3 — Current—voltage characteristics for various frequencies.

The electrical characteristics of the NLCDs indicate two important aspect for applications:

- i) the possibility to command them by low currents and voltages, compatible with those used by general semiconductor devices;
- ii) the importance of the positive temperature drift of the LCDs; this property may be used in temperature transducers or in schemes for the compensation of the negative thermal drift of other devices.

**3. The Configuration and the Characteristics of the Transducers Based on the Electro-optical Characteristics of the LCDs.** All the transducers based on the optical and/or electro-optical effects in LC are built with a light source 1 (LED or miniature bulb) a photodetector 2, a LCD 3 and a body made of an opaque material 4 (see Fig. 4). According to the purpose, the NLCD works in transmission (Fig. 4a) or in reflection mode (Fig. 4b).

The electrical characteristics of such a device are obtained from the characteristics of the LCD and of the photocoupler made of the light source and the photodetector. Typical characteristics obtained for different temperatures, using LCD 2 and a photocoupler with photoresistance are plotted in Fig. 5.

The optical transmittance of the N—LCDs and consequently the electrical characteristic of the transducer is strongly dependent on the frequency of the applied signal. The experimental characteristics plotted in Figs. 6a and 6b, for LCD 1 and LCD 2 respectively, prove the above mentioned property.

**4. The Choice of the Electrical Pre-excitation.** The electro-optical characteristic of the LCDs may be used for the detection or measuring of the electrical parameters such as voltages, currents, frequencies, difference of two frequencies or for measuring forces, displacements and temperatures.



The choice of the electrical pre-excitation is determined by the necessity of obtaining the best sensitivity on the un-linear characteristic of the measuring device.

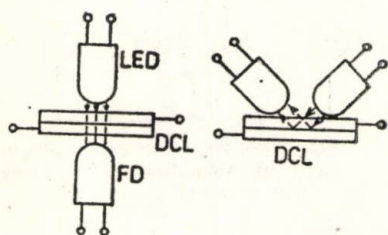


Fig. 4 — The usual configurations of the transducers with NLC: a) working by transmission; b) working by reflection.

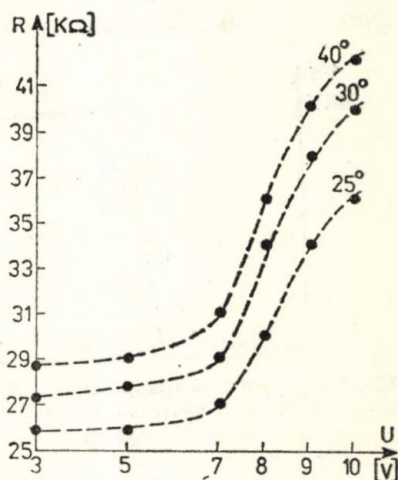


Fig. 5 — Typical characteristics of the transmission configuration using LCD 2 for different temperatures.

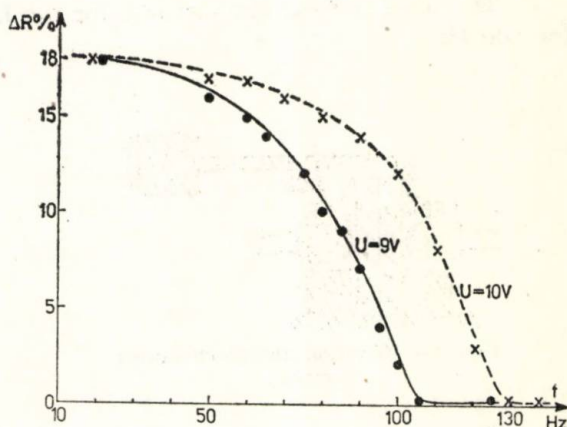
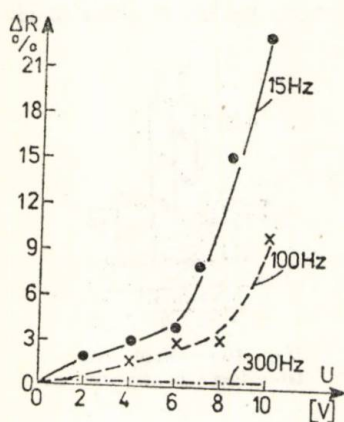


Fig. 6 — Characteristics of the transducer for different frequencies: a) using LCD 1; b) using LCD 2.

As a general rule, for measuring devices it is suitable to use the devices in the linear regime, while the sensing devices must be pre-excited by an electrical voltage a little lower than the threshold voltage  $U_p$  (see Fig. 7). This rule is exemplified below in some applications.

The detection of a voltage greater than a given value  $U_0$  may be performed with a NLCD pre-excited by a voltage equal to the difference  $U_p - U_0$ . The sensing precision is determined by the gradient of the electro-

optical characteristic, for the above application being used LCDs presenting a sharp threshold. High contrast devices are indicated.

Connecting some LCDs in parallel, each of them having a threshold voltage lower than the last, signals of low frequency may be measured. Such a device, known as „voltage rule“, is used without pre-excitation.

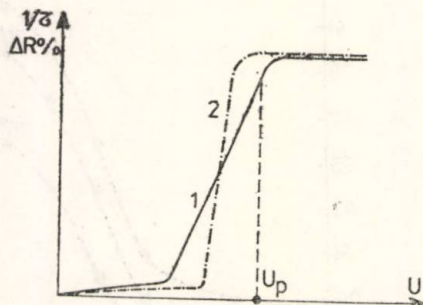


Fig. 7 — Typical characteristics of the N-LCDs used in detection (1) and in measuring (2) regime.

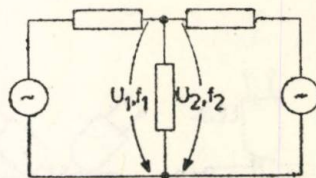


Fig. 8 — Experimental arrangement for evaluating the frequency difference of two signals.

When constant voltage  $U_1$  signals are to be controlled, the detection of a frequency lower than a given value  $f < f_0 < 200$  Hz may be performed using the electro-optical characteristic and using a pre-excitation signal which satisfies the condition  $U_0 = U_p(f_0) - U_1$ .

The sensitivity of the method, for the LCDs used by us, is about 5 Hz for 100 Hz.

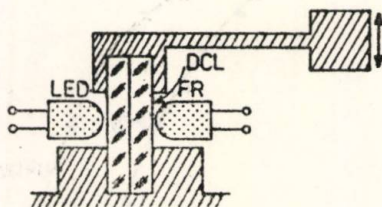


Fig. 9 — Vibration transducer configuration.

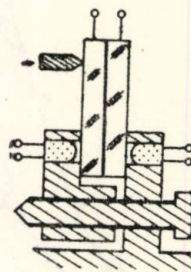


Fig. 10 — Displacement transducer configuration.

The measurement of the difference of two low and comparable frequencies may be performed visualising the interference product by means of a LCD as indicated in Fig. 8. The voltage values must satisfy the following conditions:  $U_1, U_2 < U_p$ ;  $U_1 + U_2 > U_p$ .

The electro-optical effect consists in the state commutation of a LCD with the beat frequency. The frequency may be measured by means of a chronometer if available, or the optical information itself may be good enough, if not. By this method, differences of 0,1 Hz may be pointed out in frequency ranges up to 150 ... 200 Hz. The recommended voltage values are  $U_1 = U_2 = 0,7 U_p$ ; if the amplitude of one signal is lower than  $0,7 U_p$ ,



then  $U_2 = U_p (U_1 + U_2 - 1,5 U_p)$ . If  $|f_1 - f_2| > 20$  Hz, the LCD will remain in the optical-active state and it could be used as a detector of frequency deviation of more than 20 Hz. A 50 Hz standard generator may be used in order to measure, by this method, the electrical network frequency.

Configurations of special devices might be used to detect small displacements or vibrations as shown in Fig. 9 and Fig. 10 respectively. In these cases, the optical effect is due both to the electrical excitation and to the mechanical distortion forces. The best sensitivity to the mechanical parameter is to be obtained for a pre-excitation voltage of the LCD just a little lower than the threshold voltage.

**5. Conclusions.** The use of the DD-N-LCDs in transducers for electrical and un-electrical parameters is based on specific electro-optic or electric effects. The suitable choice of the transducer configuration, of the LCD and of suitable pre-excitation allow the obtaining of the best sensitivity in the measuring or in the detecting regime.

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#### ALEGEREA PRE-EXCITAȚIEI DISPOZITIVELOR CU CRISTALE LICHIDE UTILIZATE CA TRADUCTORI

(Rezumat)

Traductoarele discutate în lucrare se bazează pe caracteristicile electrice și electro-optice ale dispozitivelor cu cristale lichide nematice cu difuzie dinamică, utilizate ca atare sau intercalate într-un fotocuplor. Rezultatele experimentale, obținute cu două dispozitive cu proprietăți diferite, evidențiază posibilitatea măsurării tensiunilor, frecvențelor, a diferențelor de frecvență a semnalelor electrice precum și a deplasărilor mici și temperaturilor. Utilizarea cu maximă sensibilitate a acestor dispozitive impune alegerea, în funcție de aplicație, a valorii pre-excitației electrice atât ca amplitudine cât și ca frecvență.

THE UNIVERSITY OF CHICAGO  
DIVISION OF THE PHYSICAL SCIENCES  
DEPARTMENT OF PHYSICS  
CHICAGO, ILLINOIS 60637  
U.S.A.  
JANUARY 1964  
TO THE DIRECTOR, NATIONAL BUREAU OF STANDARDS  
WASHINGTON, D.C. 20535  
FROM: DR. ROBERT H. DICK, JR.  
SUBJECT: A PROPOSAL FOR A JOINT PROJECT  
ON THE THEORY OF THE ELECTROMAGNETIC  
INTERACTION OF PARTICLES AND FIELDS  
The purpose of this proposal is to describe a joint project  
between the Department of Physics at the University of Chicago  
and the National Bureau of Standards. The project is intended  
to be a collaborative effort between the two institutions  
in the field of the theory of the electromagnetic interaction  
of particles and fields. The project is intended to be a  
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of the theory of the electromagnetic interaction of particles  
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1. INTRODUCTION  
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## INVESTIGAȚII ELECTRONICE PRIVIND UNELE ASPECTE MORFOLOGICE ALE FIRELOR ȘI FOLIILOR DE POLIETILENTEREFTALAT

DE

SOFIA MELINTE, M. LEANCĂ, MARCEL MOISE și N. MATEESCU

**1. Introducere.** Interesul actual în investigațiile morfologice ale polimerilor este dictat în special de dependența unor proprietăți importante, nu numai de structura chimică a polimerului ci și de cea supramoleculară. În general studiile morfologice implică mai mult metode fizice ca : difracția de raze X, analiza termică diferențială, dicroism, măsurători de densitate, cît și microscopie optică și electronică. Numai prin folosirea tuturor metodelor se poate pune în evidență structura atît de complexă a polimerilor înalți.

Cercetarea polimerilor prin diverse metode are drept scop elucidarea structurii de bază a formațiunilor morfologice supramoleculare cît și stabilirea unei corelații între acestea și procesele tehnologice de obținere a firelor și foliilor. Cunoscînd această corelație se creează posibilitatea indicării condițiilor în care se pot obține anumite structuri bine determinate, deci fibre și folii cu anumite proprietăți fizico-mecanice.

Deoarece structura firelor și foliilor de polimer este foarte complexă, depinzînd de modul lor de obținere, rezultatele obținute de diverși cercetători diferă între ele, dar nu sînt contradictorii ci se completează.

Pe lîngă alte tehnici folosite în studiul atît de complex al polimerilor filabili, microscopia electronică are avantajul că vizualizează direct structura internă a unei fibre, folosind tehnica secțiunilor sau punerea în evidență a imaginii structurale a fibrei sau foliei prin metoda replicilor.

**2. Tehnica experimentală.** Pentru a pune în evidență structura internă a unei fibre sau folii de polietilentereftalat este necesar ca prin diverse tehnici de corodare să se pătrundă în structura intimă a polimerului fără a o modifica. În acest scop sînt cunoscute corodările cu neutroni, cu ultrasunete, chimic, în azot lichid și în plasmă. Funcție de structura polimerului se folosește și tehnica de evidențiere a structurii lui interne. Pentru PET se pare că cele mai bune rezultate se obțin prin corodarea crioscopică, chimică și în plasmă.

În scopul vizualizării directe a structurii interne a unei fibre de PET s-au efectuat secțiuni transversale cu ajutorul unui ultramicrotom tip LKB III. Grosimea secțiunilor a variat în jurul  $6 \dots 8 \cdot 10^{-2} \mu\text{m}$ . Pentru a obține



un contrast corespunzător, în vederea evidențierii elementelor structurale interne, cu ajutorul microscopului electronic tip TESLA BS 613—A, secțiunea a fost în prealabil nuanțată cu paladiu, în vid, sub un unghi de  $19...20^\circ$ .

Pentru obținerea unor rezultate de profunzime privind structura fibrelor s-a folosit corodarea chimică în vapori de acid sulfuric timp de  $45'..60'$ . De pe fibra astfel corodată se lua replica cu ajutorul unei pelicule de colodiu sau polistiren, pe care (după înlăturarea fibrei), se depunea un strat de cărbune, în vid ( $\sim 10^{-6}$  mm Hg), pe direcție normală la pelicula de colodiu, apoi se nuanța cu Ag sub un unghi de  $19...20^\circ$  pentru a obține un contrast cât mai bun. Replica obținută pe pelicula de colodiu se taie în pătrate de  $2 \times 2$  mm și se introduce în apă distilată. După un timp de  $3...12$  ore se desprinde la suprafața apei pelicula de C—Ag, care se așază pe grilă, se usucă și se studiază la microscopul electronic.

Dacă replica se ia pe o peliculă de polistiren, aceasta se solvă în cloriform, obținând în final replica de C—Ag.

Pentru folii de PET s-a folosit și corodarea chimică cu KOH, 30% în apă, la temperatura camerei.

În scopul evidențierii diverselor detalii de suprastructură în foliile de PET, s-a folosit metoda corodării în plasmă care în cazul acestui polimer duce la rezultate concludente. Parametrii caracteristici ai generatorului de plasmă utilizat au fost: presiunea de activare a argonului  $2.10^{-3}$  torr, frecvența 30 MHz cu o intensitate a curentului de 120 mA la o tensiune de 50 V. Timpul de corodare, 30 minute.

**3. Rezultate și discuții. 3.1. Prezentarea rezultatelor.** În fig. 1 este redată secțiunea transversală a unui monofilament de PET etirat. Replica suprafeței unui filament filat este prezentată în fig. 2.

Microfotografia din fig. 3 pune în evidență suprastructura filamentului de PET supus etirării.

Pentru a scoate în evidență mecanismul de formare a aceleiași structuri atât pentru fibre cât și pentru filme (folii) de PET, s-au studiat în paralel ambele tipuri, fibre și folii. Replica suprafeței foliei de PET extrusă este redată în fig. 4. iar efectul monoetirării asupra foliei de PET extrusă este redat în fig. 5.

Folia monoetirată supusă unui proces de etirare pe o direcție perpendiculară la prima etirare, adică folia bietirată, are o structură deosebită de precedentă, evidențiată prin microscopie electronică (fig. 6).

**3.2. Discuția rezultatelor prezentate pe baza modelelor existente.** Pentru a interpreta rezultatele obținute credem că ar fi indicat să prezentăm diversele modele imaginate de diferiți autori din necesitatea de a explica rezultatele experimentale obținute, cât și neconcordanța unora din ele cu mecanismul de formare a fibrelor și foliilor sintetice din polimeri semicristalini.

Deși acest domeniu al polimerilor înalți și de obținere a fibrelor și foliilor sintetice, este relativ nou, aproximativ 40 ani, sînt numeroase studii, în special bazate pe microscopie electronică, care evidențiază mecanismul de formare a structurii interne a acestora. Aceste rezultate experimentale trebuiau explicate teoretic, pentru a pune la dispoziția fabricantului de fibre, tehnologia de fabricație, modul de a acționa asupra polimerului pentru a obține fibre cu caracteristici fizice superioare.





Fig. 1 — Microfotografia unei secțiuni a filamentului de PET monocetirat.

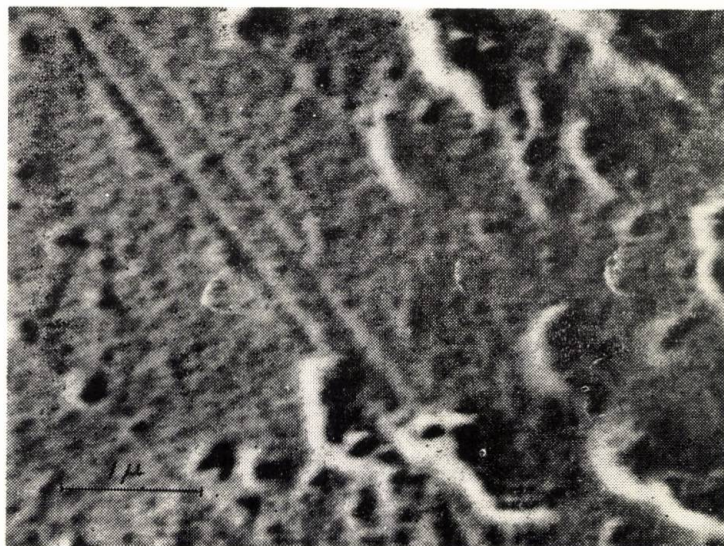


Fig. 2 — Replica suprafeței unui filament de PET filat.

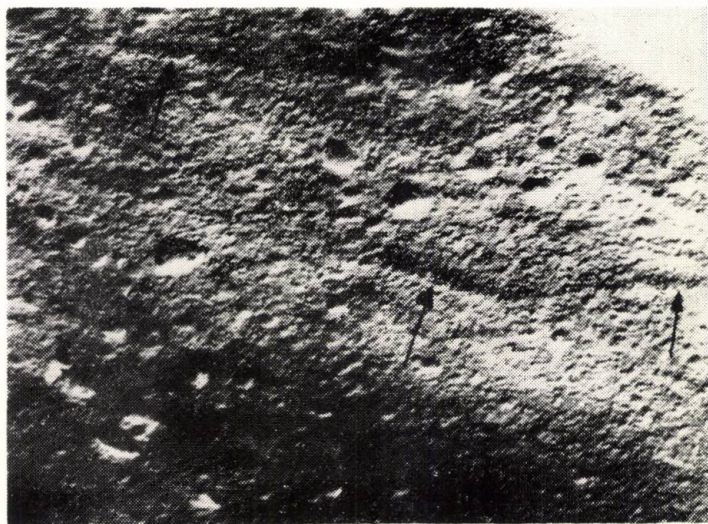


Fig. 3 — Microfotografia unui filament de PET etirat.

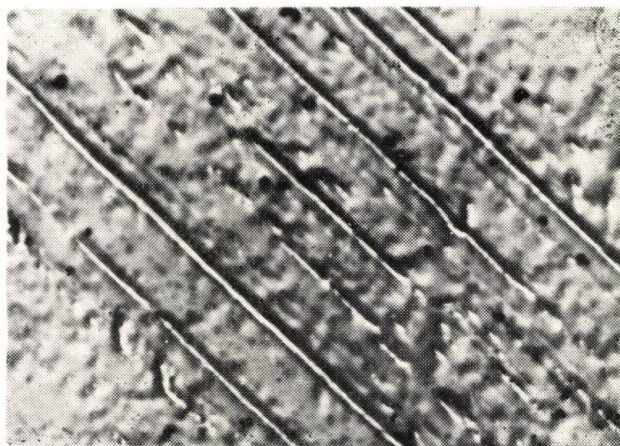


Fig. 4 — Replica suprafeței foliei de PET extrusă.



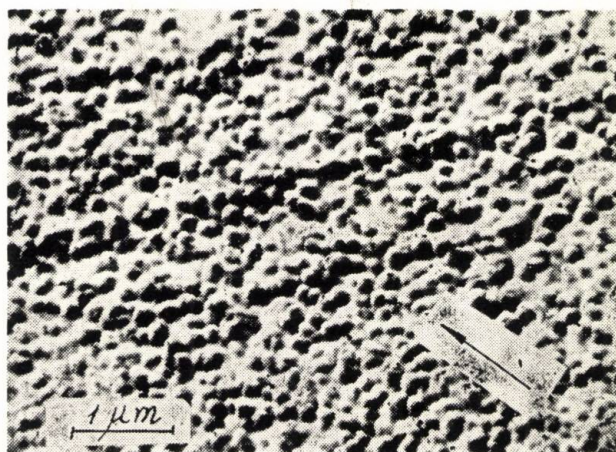


Fig. 5 — Replica suprafeței foliei de PET monoextrusată.

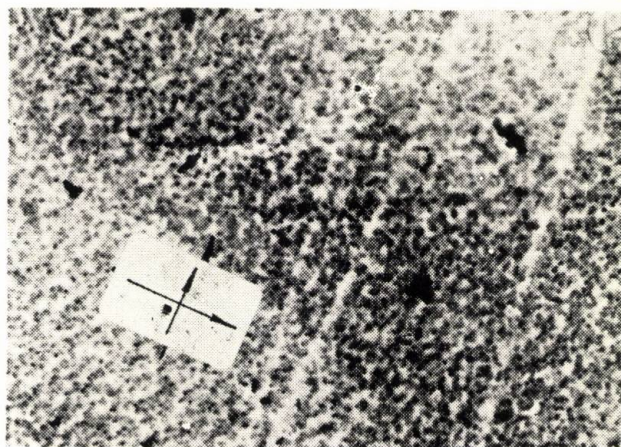
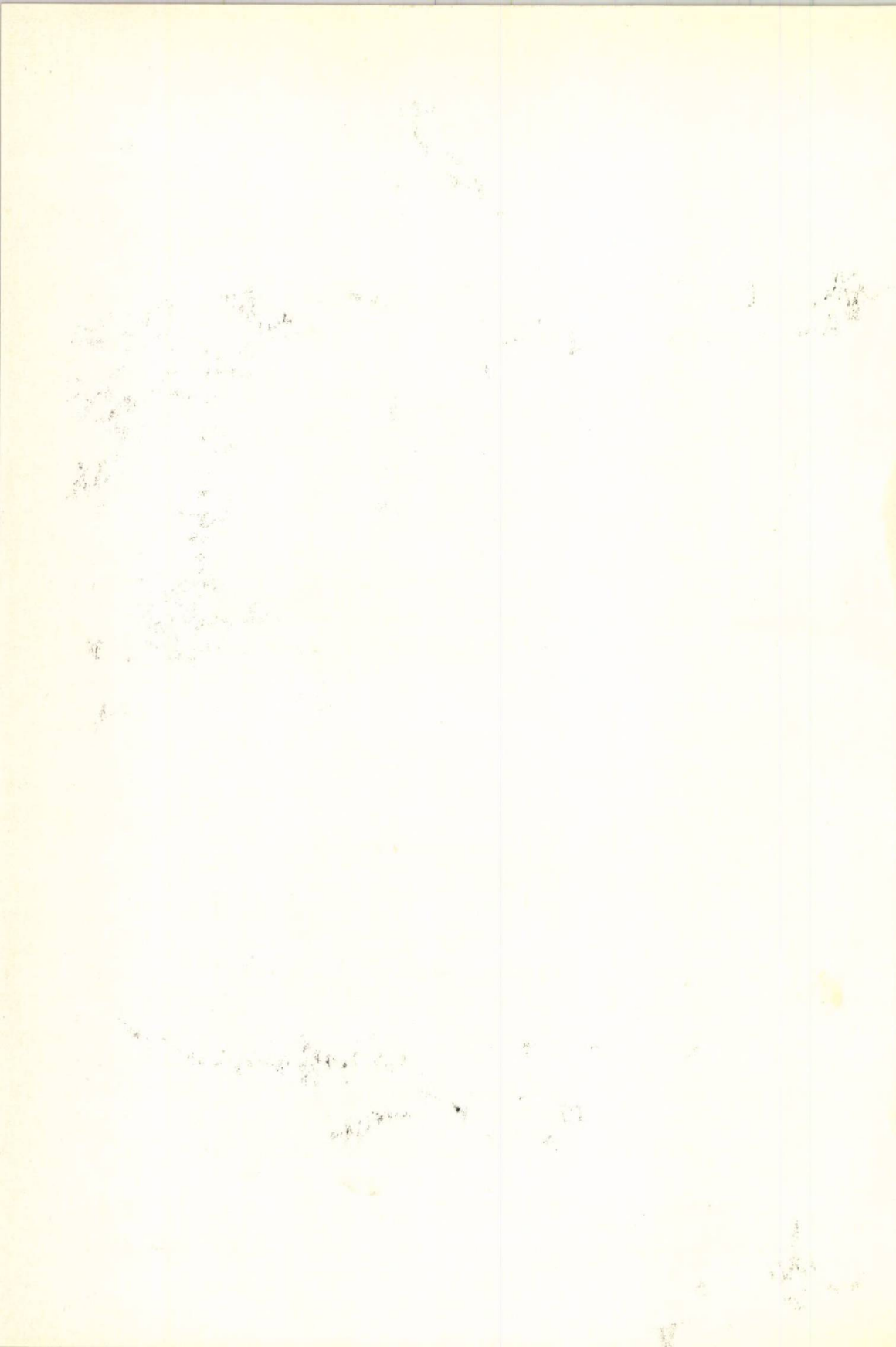


Fig. 6 — Replica suprafeței foliei de PET bietirată.





Din literatura de specialitate este cunoscut faptul că fibrele sintetice ca și fibrele naturale au o structură moleculară caracteristică. În polimerii înalți liniari, lanțurile moleculare variază de la câteva sutimi de angströmi la câteva milionimi. Aranjamentul acestor lanțuri macromoleculare între ele duce la formarea microfibrilelor care variază între câteva zecimi până la câteva

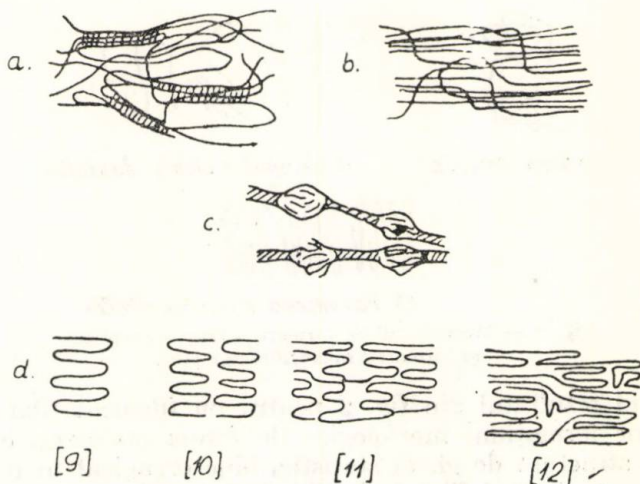


Fig. 7 — Diverse modele clasice privind mecanismul de formare al fibrelor sintetice: a) micelar franjulat [5]; b) fibrilar franjulat [7]; c) domenii cristaline și amorse [8]; d) model pliat și depliat al lanțului molecular [9] ... [12].

sutimi de angströmi grosime; prin juxtapunerea acestor microfibrile se obțin macrofibrile sau fibre vizibile prin microscopie optică. Acestea nu sînt altceva decît fibrele textile. Acest mod de a explica structura fibrelor sintetice obținute din topitură prin filare și etirare nu corespunde realității. Studiul morfologiei fibrelor sintetice definite clasic (lanț macromolecular, microfibrile, macrofibrile și fibre) sugerează modelele de structură imaginate de diverși autori: Gerngross, Hermann și Abitz [5] (micelar franjulat), Hess și Kiessing [6], Hearle [7] (fibrilar franjulat), Statton [8] (lanțurile macromoleculare etirate duc la formarea domeniilor amorse și cristaline) (fig. 7).

Un alt grup de cercetători arată că cristalii de polipropilenă se formează prin plierea lanțurilor macromoleculare și pe baza rezultatelor obținute propun un model de structură în care lanțurile macromoleculare se pliază în acordeon: Keller [9], Stuart [10], Hoseman [11], Peterlin [12].

Toate aceste modele nu pot explica rezultatele experimentale obținute cît și în special modificările ce intervin în procesul filării, etirării și tratamentului termic. Szaboies [13] încearcă să explice mecanismul de formare al fibrelor sintetice utilizînd modelele structurale [9] ... [12] prin sintetizarea lor, reducîndu-le la două structuri ale lanțului molecular: sub formă de ghem statistic și lanț repliat în acordeon (fig. 8). El arată că trecerea de la o structură la alta reprezintă mecanismul de formare a filamentului. Progresul realizat în microscopia electronică scoate în evidență diverse detalii de structură, care nu pot fi explicate pe baza modelelor enumerate. Această neconcordanță între teorie și practică a dus la elaborarea a noi modele mai complete pe baza cărora se pot explica detaliile de structură evidențiate de microscopia electronică.

Mecanismul probabil de formare a filamentelor sintetice se poate explica mai bine pe baza modelului de structură nodulară a lanțului molecular.

În studiile pe care le-am întreprins asupra filamentelor și foliilor poliesterice [15] ... [17] în speță PET și polipropilenă (PP) precum și în această lucrare, s-a remarcat prezența unei structuri nodulare, rezultat



în acord cu cel evidențiat de Davidovitz [18], Van Veld Morris și Billica [19] pentru poliamidă, poliester și poliacrilonitril, precum și cel remarcat de Hoffman [20] pentru polimerii liniari amorfii, poliuretan și polibutadin.

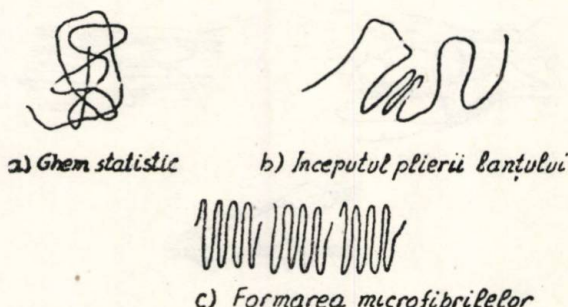


Fig. 8.— Mecanismul de formare a lanțului replicat și formarea microfibrilelor [13].

Analizînd rezultatul din fig. 2 pentru un filament filat se constată existența unor formațiuni morfologice de forma nodulară, care nu sînt altceva decît structura de ghem statistic, bine organizat în topitură, care se regăsește în polimerul filat (solidificat). Aceeași structură se regăsește și în folia de PET extrusă (fig. 4), adică formațiuni nodulare ce formează domeniiordonate, bine organizate, de dimensiuni mici și domenii intranodulare, care au ponderea cea mai mare și care constau dintr-o împachetare întîmplătoare a lanțului molecular. Liniile paralele sînt urmele lăsate de valțuri.

Mecanismul esențial în formarea filamentului sintetic constă în diferența ce există între volumul ocupat de așa numitul ghem (nodul) statistic în polimerul sub formă de topitură și cel în stare solidă, care practic poate fi determinat prin măsurarea diferenței de densitate între cele două stări.

Prin microscopie electronică, tehnica replicilor, se poate evidenția structura fibrilară a unui monofilament etirat (fig. 3) în care fibrilele sînt indicate prin săgeți.

Schema de formare a microfibrilelor și apoi a fibrilelor este dată în fig. 9 după Davidovitz [14], pe baza rezultatelor obținute de Siemens [21] și Jenckel [22] pentru poliamidă, care au scos în evidență o distribuție radială a microfibrilelor începînd de la suprafață spre interior.

Filamentul ce iese din filieră este supus unui proces de solidificare și apoi prin tratament termic unui proces de cristalizare. Prin acest proces se micșorează volumul nodulului statistic formîndu-se o cristaliță bine de-

finită a cărei axă  $\vec{c}$  este în general normală la axa fibrilei. Aranjarea acestor cristalițe, în procesul de cristalizare și etirare, formează microfibrilele, între care apar domenii suficient de mari, amorfie, numite micropori sau spații intergranulare care și ele se orientează în timpul procesului de etirare. Axa mică a cristaliței  $\vec{a}$  este normală la axa fibrilei și paralelă cu axa filamentului. Axa cristalografică  $\vec{b}$  a aranjamentului lateral al crista-



itelor, pentru formarea microfibrilelor, este paralelă la axa fibrilei și aranjată radial în raport cu filamentul (fig. 9).

În timpul etirării, aceste zone solidificate, cristalitele ce formează microfibrilele radiale, tind să se orienteze paralel cu direcția de etirare, deci trec din poziția radială în poziția paralelă cu axa filamentului, pentru rapoarte de etirare mai mari de 4:1; această orientare este completă (fig. 10 c).

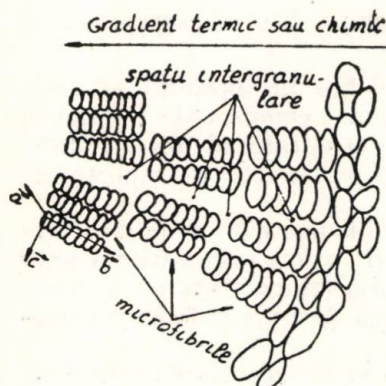


Fig. 9 — Schema de formare a microfibrilelor [14].

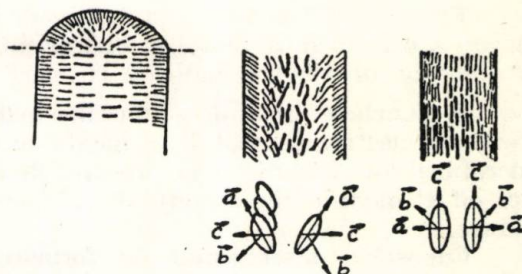


Fig. 10 — O secțiune într-un polimer extrus; a) liniile radiale sunt direcțiile microfibrilelor din fig. 9; b) în timpul etirării microfibrilele se orientează astfel încât axa  $\vec{c}$  a cristalitei are tendința de a deveni paralelă cu direcția de etirare; c) în cazul unei orientări complete, axa  $\vec{c}$  a cristalitei este paralelă cu direcția de etirare.

Același fenomen, explicat schematic în fig. 9 și 10, pentru filamentele sintetice și pus în evidență prin replica din fig. 3, dar care nu redă suficient de clar acest mecanism, se observă foarte bine în cazul foliei de PET monoetirat (fig. 5), cu singura deosebire că în acest caz nu mai avem de-a face cu o distribuție radială ci cu una planară, în planul foliei. De asemenea, domeniile nodulare au în acest caz formă globulară. Pentru a elucidă rezultatele redată în fig. 5 vom transpune mecanismul de formare al fibrilelor din cazul filamentelor sintetice la foliile de PET, redat schematic în fig. 11. Săgețile din fig. 11 indică direcția de etirare.

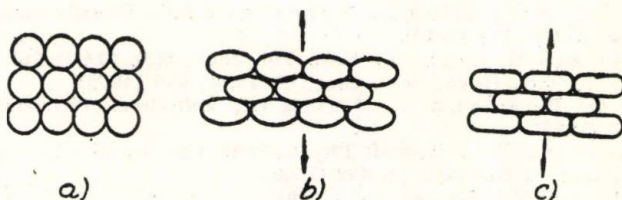


Fig. 11 — Diagrama de orientare a domeniilor globulare în timpul etirării într-o secțiune transversală: a) formarea globulelor; b) orientarea globulelor în timpul etirării; c) de-formarea acestora în timpul procesului de etirare [21].

Revenind la fig. 5, observăm că în procesul de monoetirare are loc o împachetare compactă a domeniilor globulare formate prin solidificare și tratament termic încît domeniile intragranulare sau microporii devin

foarte mici. Se observă de asemenea o tendință de orientare a axei mici a domeniului în direcția de tragere, iar a axei mari, normal la axa de tragere. În fig. 5 nu se observă deformarea domeniilor globulare, întrucât raportul de etirare nu depășește 4 : 1, când apare acest fenomen (pus în evidență în cazul PET [21] pentru rapoarte de etirare 4 : 1); dar se observă foarte clar structura globulară, precum și tendința de orientare a acestor domenii globulare după o direcție normală la direcția de tragere indicată de săgeată.

Folia de PET supusă unui proces de bătire (fig. 6) duce la o uniformizare a orientării, datorată dublei orientări, după direcții perpendiculare la direcțiile de tragere, indicate cu săgeți.

**4. Concluzii.** Studiile polimerilor înalți, filabili din topitură, prin microscopie electronică scot în evidență mecanismul complex de formare a microfibrilelor și fibrilelor în procesul de filare și etirare, al fibrelor sintetice cât și modificările structurale ale acestora în funcție de raportul de etirare.

Cunoscând mecanismul de formare a filamentelor și foliilor sintetice se poate modifica tehnologia de fabricație a lor, prin varierea unor parametri ca : raport de etirare, viteza de filare, temperatura de termofixare, etc., obținând în acest fel o gamă largă de fibre și folii cu caracteristici deosebite.

Tehnica electronică de studiu a structurii supramoleculare completată cu celelalte tehnici amintite, poate pune în evidență structura cât și modificările morfologice ce apar în timpul diverselor procese tehnologice de obținere a firelor și foliilor.

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#### ELECTRONIC INVESTIGATIONS CONCERNING SOME MORPHOLOGICAL ASPECTS OF POLYETHYLENETEREPHTALAT (PET) FIBRES AND FILMS

(Summary)

With a view to visualize PET morphological formations by means of electronic microscopy, the section and replica technique was used.

In order to go into the core of the polymer, the latter was subjected to chemical or in-plasma corrosion. Plasma corrosion constitutes one of the modern methods designed to reveal PET morphology. It should be pointed out that replicas obtained show polymer internal structure which changes during the process of spinning and stretching.

Formations of globular type were noticed in the case of spinned fibres which change into fibrillar formations during the process of stretching. Similarly, in the case of extruded films, the formations are granular, alternating with intergranular domains which orientate during stretching in a normal direction to that of stretching. The same orientation phenomenon is revealed in the case of double stretched PET film; yet, being doubly orientated, it appears less orientated, leaving the impression of uniformization.





## ANALYSES ROENTGENOSTRUCTURELLES CONCERNANT LE SYSTÈME $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$

PAR

NASTASIA VAASILIU et MIHAI-LIVIU CRAUS

Les systèmes des ferrites mixtes contenant aussi du Ca imposent la détermination des conditions où les ions de Ca, ayant l'un des plus grands rayons ioniques (1,06 Å [1]), peuvent se substituer dans le réseau de spinelle aux ions ayant des rayons ioniques plus petits, comme les ions de Fe (0,82 Å), Zn (0,83 Å), Ni (0,78 Å) et d'autres, avec la formation d'un composé à structure de spinelle pur, monphasé ou d'un composé pluriphasé.

Jusqu'à présent on a obtenu les ferrites mixtes  $\text{Ca}_x\text{Zn}_{1-x}\text{Fe}_2\text{O}_4$  [2] les calciomagnétites  $\text{Ca}_x\text{Fe}_{3-x}\text{O}_4$  [3] ... [9] ainsi que les ferrites de Mn—Zn [10] ... [14] auxquels, dans des conditions stœchiométriques, on a ajouté du Ca jusqu'à 0,1 % mol. La structure du ferrite  $\text{Ca}_x\text{Zn}_{1-x}\text{Fe}_2\text{O}_4$  a été déterminée par des analyses métallographique et celle du calciomagnétite par des analyses roentgenostructurelles, mesurant l'aimantation spontanée et étudiant l'effet Mössbauer.

Au cas du ferrite de Mn—Zn avec addition de Ca la détermination de la localisation du Ca (dans le réseau de spinelle, par la substitution des ions de Zn, ou aux joints des cristallites, par ségrégation) a été réalisée par le dosage chimique fractionné du Ca, par l'utilisation des ions de Ca radioactif et par diffraction de rayons X.

Voici les résultats de ces recherches :

a) les ions de Ca, qui introduisent une forte énergie de déformation, peuvent rester dans le réseau en se substituant aux ions métalliques du sousréseau tétraédrique. Cela, généralement, du cas d'un refroidissement rapide (trempe dans l'air ou dans l'eau) de la température de frittage à la température de la chambre. Au cas d'un refroidissement lent une partie des ions de Ca ségrègent aux joints des cristallites ;

b) la quantité de Ca qui pénètre dans le réseau de spinelle, en se substituant à une partie des ions métalliques du sousréseau tétraédrique, formant un ferrite à structure de spinelle, pur, monphasé, est limitée, elle ne dépasse pas la valeur de  $x \simeq 0,30$  au cas d'un refroidissement rapide et la valeur de  $x \simeq 0,2$  au cas d'un refroidissement lent. Pour des quantités de Ca plus grandes, le composé obtenu est pluriphasé, c'est-à-dire, outre la phase spinelle apparaît aussi d'autres phases représentant des ferrites du système Ca—Fe.



Après avoir préparé le système  $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$  [15] ... [18], on a effectué aussi certaines analyses roentgenostructurelles de ce ferrite sintérisé, à  $1175^\circ\text{C}$  [19].

La température de frittage étant trop basse les résultats obtenus n'ont pas été édifiants.

On a réalisé de nouvelles séries des ferrites de  $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$  ( $x$  ayant les valeurs de 0,00 ; 0,01 ; 0,02 ; 0,05 ; 0,07 ; 0,10 ; 0,20 ; 0,25 ; 0,30 ; 0,40 ; 0,50 ; 0,60 et 0,70) présintérisés pendant une heure à la température de  $1\ 000^\circ\text{C}$  et ensuite sintérisés pendant 6 heures à la température de  $1\ 300^\circ\text{C}$  ou pendant 4 heures à la température de  $1\ 350^\circ\text{C}$ .

Après avoir été sintérisés, une partie des échantillons ont été rapidement refroidis avec une vitesse de  $800^\circ\text{C}/\text{minute}$ , une autre partie a été lentement refroidie avec une vitesse de  $1,5^\circ\text{C}/\text{minute}$  et une troisième partie a été refroidie très lentement avec une vitesse de  $0,5^\circ\text{C}/\text{minute}$ .

Les analyses roentgenostructurelles ont été réalisées à l'aide d'un diffractomètre DRON-1 en utilisant la radiation  $\text{MoK}\alpha$  filtrée et monochromatisée et aussi à l'aide de la méthode Debye-Scherrer en utilisant une installation de rayons X de type TUR M 62 et la radiation  $\text{CoK}\alpha$  filtré.

Certains diffractogrammes et debyeogrammes obtenus sont reproduits par les figures 1 ... 4 et les autres par les figures 5 ... 7.

La indexation des lignes spectrales a été réalisée à l'aide d'un comparateur IZA-2, ayant une précision de  $\pm 0,02\ \text{mm}$  ; les résultats obtenus, concernant les valeurs des distances interplanaires  $d_{hkl}$ , sont donnés dans le tableau 1, où pour avoir la possibilité de comparer on présente aussi les valeurs  $d_{hkl}$  du ferrite de  $\text{NiFe}_2\text{O}_4$ .

La constante de réseau correspondant à la phase spinelle a été déterminée par la méthode Bradley-Jay. La fig. 8 présente la variation de la constante de réseau  $a$  du ferrite en fonction de la concentration des ions de Ca,  $x$ , tant au cas des ferrites de  $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$  rapidement refroidis que des ferrites refroidis lentement.

De l'analyse des diffractogrammes et des debyeogramme, présentés par les figures 1 ... 7, de l'analyse des distances interplanaires  $d_{hkl}$  du tableau 1, de la manière dont varie la constante de réseau  $a$  fonction de concentration  $x$  dans les ions de Ca (fig. 8) on peut faire les remarques suivantes.

Tant les diffractogrammes présentés dans les figures 1 et 2 correspondant au ferrite de  $\text{Ca}_{0,15}\text{Ni}_{0,85}\text{Fe}_2\text{O}_4$  sintérisé à  $1\ 350^\circ\text{C}$  rapidement ou lentement refroidi, que les debyeogrammes présentés dans les figures 5 et 6 correspondants aux ferrites  $\text{Ca}_{0,02}\text{Ni}_{0,98}\text{Fe}_2\text{O}_4$  et  $\text{Ca}_{0,2}\text{Ni}_{0,8}\text{Fe}_2\text{O}_4$  sintérisés à  $1\ 300^\circ\text{C}$  refroidis rapidement et les valeurs des distances interplanaires  $d_{hkl}$  obtenues, démontrent que les ferrites en question sont des composés à structure des spinelles purs, monophasés, c'est-à-dire les composés du système  $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$  (où l'on tient compte des valeurs indiquées) sintérisés à  $1\ 300^\circ\text{C}$  ou  $1\ 350^\circ\text{C}$  et refroidis rapidement ou même lentement, sont des composés monophasés.

Le diffractogramme de la fig. 3, correspondant au composé  $\text{Ca}_{0,2}\text{Ni}_{0,8}\text{Fe}_2\text{O}_4$  sintérisé à  $1\ 350^\circ\text{C}$ , mais très lentement refroidi (avec une vitesse de refroidissement de  $0,5^\circ\text{C}/\text{minute}$  et avec un palier de 6 heures à la température de  $1\ 100^\circ\text{C}$ ), de même que les valeurs obtenues pour les distances interplanaires  $d_{hkl}$  données dans le tableau 1, montre que cette



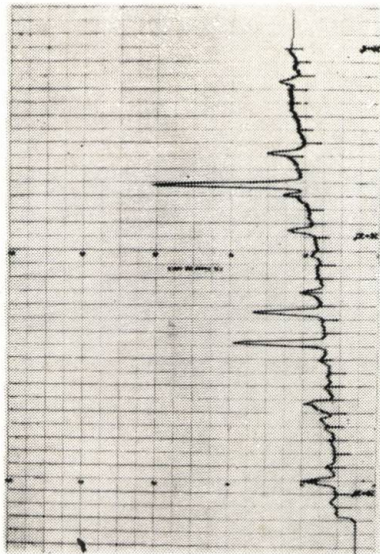


Fig. 1 — Ferrite  $\text{Ca}_{0.15}\text{Ni}_{0.85}\text{Fe}_2\text{O}_4$  sintérizé à  $1350^\circ\text{C}$  et rapidement refroidi.

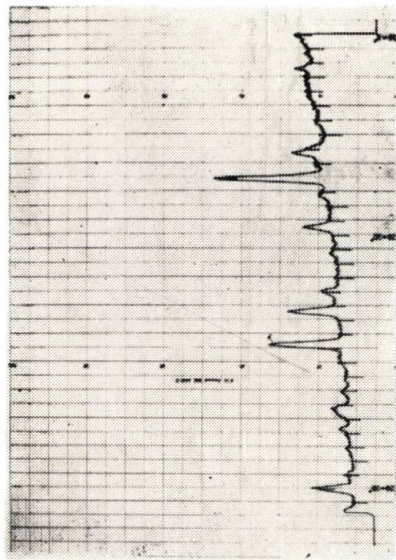


Fig. 2 — Ferrite  $\text{Ca}_{0.15}\text{Ni}_{0.85}\text{Fe}_2\text{O}_4$  sintérizé à  $1350^\circ\text{C}$  et lentement refroidi.

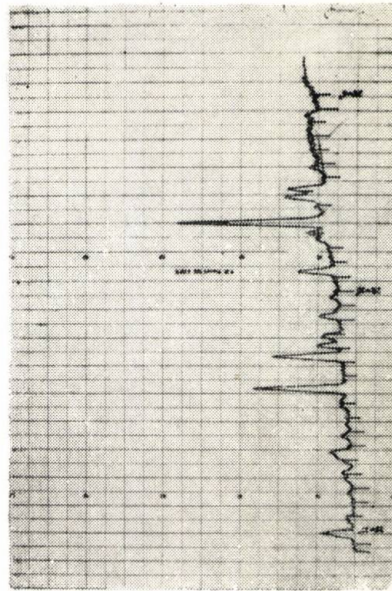


Fig. 3 — Ferrite  $\text{Ca}_{0.2}\text{Ni}_{0.8}\text{Fe}_2\text{O}_4$  sintérizé à  $1350^\circ\text{C}$  et très lentement refroidi avec un palier de 6 heures à la température de  $1100^\circ\text{C}$ .

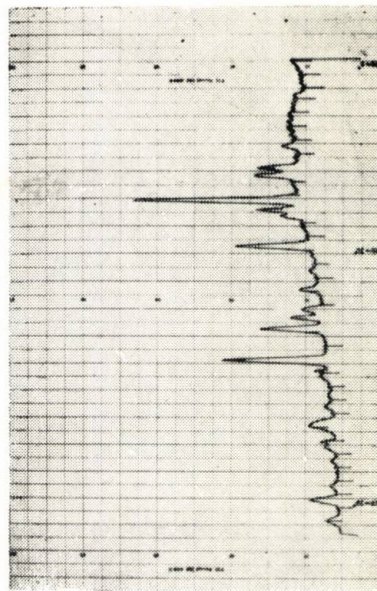


Fig. 4 — Ferrite  $\text{Ca}_{0.4}\text{Ni}_{0.6}\text{Fe}_2\text{O}_4$  sintérizé à  $1350^\circ\text{C}$  et lentement refroidi.



Fig. 5 — Ferrite  $\text{Ca}_{0,76}\text{Ni}_{0,98}\text{Fe}_2\text{O}_4$  sintérisé à  $1300^\circ\text{C}$  et lentement refroidi.

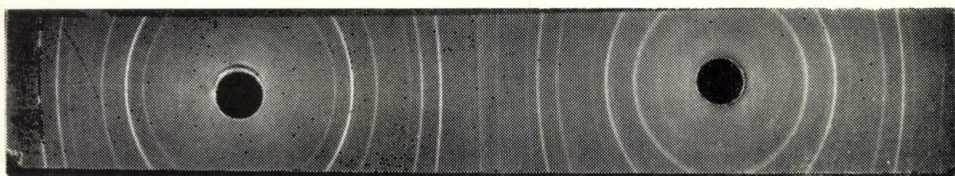


Fig. 6 — Ferrite  $\text{Ca}_{0,2}\text{Ni}_{0,8}\text{Fe}_2\text{O}_4$  sintérisé à  $1300^\circ\text{C}$  et rapidement refroidi.



Fig. 7 — Ferrite  $\text{Ca}_{0,7}\text{Ni}_{0,3}\text{Fe}_2\text{O}_4$  sintérisé à  $1300^\circ\text{C}$  et rapidement refroidi.



Tableau 1  
Les distances interplanaires  $d_{hkl}$

| Ferrite $\text{NiFe}_2\text{O}_4$<br>fiche 8100<br>$T_s = 1350^\circ\text{C}$ | Ferrite $\text{Ca}_{0,15}\text{Ni}_{0,85}\text{Fe}_2\text{O}_4$<br>rapidement<br>refroidi<br>$T_s = 1350^\circ\text{C}$ | Ferrite $\text{Ca}_{0,15}\text{Ni}_{0,85}\text{Fe}_2\text{O}_4$<br>lentement refroidi<br>$T_s = 1350^\circ\text{C}$ | Ferrite $\text{Ca}_{0,02}\text{Ni}_{0,98}\text{Fe}_2\text{O}_4$<br>rapidement<br>refroidi<br>$T_s = 1300^\circ\text{C}$ | Ferrite $\text{Ca}_{0,2}\text{Ni}_{0,8}\text{Fe}_2\text{O}_4$<br>rapidement<br>refroidi<br>$T_s = 1300^\circ\text{C}$ |
|-------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------|
| 4,820                                                                         | 4,832                                                                                                                   | 4,841                                                                                                               | 4,700                                                                                                                   | 4,680                                                                                                                 |
| 2,948                                                                         | 2,927                                                                                                                   | 2,955                                                                                                               | 2,918                                                                                                                   | 2,918                                                                                                                 |
| 2,513                                                                         | 2,531                                                                                                                   | 2,533                                                                                                               | 2,488                                                                                                                   | 2,497                                                                                                                 |
| 2,408                                                                         | 2,392                                                                                                                   | 2,398                                                                                                               | 2,388                                                                                                                   | 2,392                                                                                                                 |
| 2,085                                                                         | 2,098                                                                                                                   | 2,107                                                                                                               | 2,071                                                                                                                   | 2,089                                                                                                                 |
| 1,912                                                                         | 1,922                                                                                                                   | 1,891                                                                                                               | 1,890                                                                                                                   | —                                                                                                                     |
| 1,702                                                                         | 1,713                                                                                                                   | 1,704                                                                                                               | 1,690                                                                                                                   | 1,709                                                                                                                 |
| 1,605                                                                         | 1,593                                                                                                                   | 1,619                                                                                                               | 1,595                                                                                                                   | 1,615                                                                                                                 |
| 1,476                                                                         | 1,483                                                                                                                   | 1,483                                                                                                               | 1,466                                                                                                                   | 1,487                                                                                                                 |
| 1,410                                                                         | 1,416                                                                                                                   | 1,418                                                                                                               | 1,408                                                                                                                   | 1,409                                                                                                                 |
| 1,318                                                                         | 1,328                                                                                                                   | 1,328                                                                                                               | 1,312                                                                                                                   | 1,320                                                                                                                 |
| 1,270                                                                         | 1,280                                                                                                                   | 1,278                                                                                                               | 1,265                                                                                                                   | 1,275                                                                                                                 |
| 1,257                                                                         | —                                                                                                                       | —                                                                                                                   | 1,255                                                                                                                   | 1,258                                                                                                                 |
| 1,203                                                                         | 1,215                                                                                                                   | 1,214                                                                                                               | 1,194                                                                                                                   | 1,205                                                                                                                 |
| 1,167                                                                         | 1,180                                                                                                                   | 1,172                                                                                                               | 1,164                                                                                                                   | 1,169                                                                                                                 |
| 1,114                                                                         | 1,118                                                                                                                   | 1,118                                                                                                               | 1,108                                                                                                                   | 1,117                                                                                                                 |
| 1,085                                                                         | 1,092                                                                                                                   | 1,097                                                                                                               | 1,081                                                                                                                   | 1,088                                                                                                                 |
| 1,042                                                                         | 1,047                                                                                                                   | 1,047                                                                                                               | 1,039                                                                                                                   | 1,045                                                                                                                 |

fois-ci on a obtenu un composé pluriphasique, parce que à côté des lignes spectrales qui correspondent à la phase spinelle apparaissent aussi d'autres lignes spectrales correspondant à des phases étrangères comme  $\alpha\text{-Fe}_2\text{O}_3$ ,  $\text{CaO}$  ou même un ferrite du système  $\text{CaFe}$ .

Le diffractogramme de la fig. 4 correspondant au ferrite  $\text{Ca}_{0,4}\text{Ni}_{0,6}\text{Fe}_2\text{O}_4$  sintérisé à  $1350^\circ\text{C}$  et refroidi très lentement et la debyegramme de la fig. 7, correspondant au ferrite  $\text{Ca}_{0,7}\text{Ni}_{0,3}\text{Fe}_2\text{O}_4$  sintérisé à  $1300^\circ\text{C}$  et refroidi lentement, de même que les distances interplanaires correspondantes (tableau 1) montre clairement que ces composés sont pluriphasés, parce que à côté de la phase spinelle apparaissent aussi d'autres phases des ferrites du système  $\text{CaFe}$ .

Il résulte qu'entre certaines limites de la concentration  $x$  en ions de calcium et fonction de la manière de refroidissement des ferrites sintérisés — rapidement, lentement ou très lentement, — les ions de calcium à grand rayon ionique peuvent substituer dans le réseau spinelle du ferrite  $\text{NiFe}_2\text{O}_4$ , certains ions de Ni, ayant le rayon ionique beaucoup plus réduit, avec la formation du système spinelle  $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$  monophasé.

Cette conclusion apparait comme évidente en regardant la courbe de variation de la constante de réseau  $a$  en fonction de la concentration  $x$  des ions de calcium (fig. 8). On constate que jusqu'aux concentration  $x = 0,15$  les constantes de réseau des échantillons lentement ou rapidement refroidis coïncident. Pour les échantillons lentement refroidis la constante de réseau touche la valeur maximum à  $x \simeq 0,15$ , et pour  $x > 0,15$ , la valeur de celle-ci reste constante entre les limites de erreurs des mesure. Pour les échantillons

rapidement refroidis le maximum de valeur de la constante de réseau  $a$  correspond à  $x \simeq 0,30$  et pour  $x > 0,30$  la valeur de celle-ci reste presque constante.

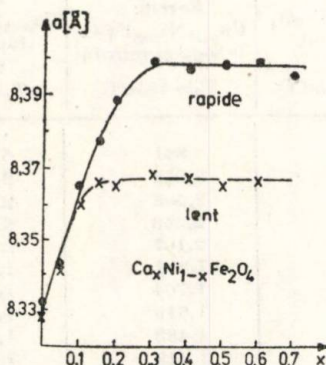


Fig. 8—Variation de la constante de réseau  $a$  avec la concentration  $x$  des ions de Ca.

Puisque la constante de réseau ne présente plus de modifications pour des valeurs qui dépassent  $x = 0,15$  (pour les ferrites lentement refroidis) et  $x = 0,30$  (pour les ferrites rapidement refroidis), il résulte que la solubilité maximum du calcium dans le ferrite  $\text{NiFe}_2\text{O}_4$  est limitée à ces valeurs. La variation de la constante de réseau montre que jusqu'aux concentrations en ions de calcium correspondantes à la valeur  $x = 0,15$ , les ferrites obtenue sont monophasés et stables, pour les hautes, de même que pour les basses températures, quelle que soit leur manière de refroidissement, rapide ou lente. Dans l'intervalle  $0,15 < x < 0,30$ , les ferrites obtenus sont certainement stables à la température de sinterisation, mais ils sont instables à des basses températures, et cela en fonction de la manière de refroidissement utilisée : rapide ou lente, ce qui conduit à l'obtention des ferrites monophasés (refroidissement rapide) ou pluriphasés (refroidissement lent).

Pour des concentration  $x > 0,30$  tous les composés du système  $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$  sont instables, pluriphasés, quelles que soit leur manière de refroidissement.

Des analyses roentgenostructurales effectuées a résulté que les phases étrangères qui apparaissent aux composés pluriphasés sont  $\text{CaO}$ ,  $\alpha\text{-Fe}_2\text{O}_3$  et un composé  $\text{Ca-Fe}$  qui peut être  $\text{CaFe}_2\text{O}_4$  ou  $\text{Ca}_2\text{Fe}_2\text{O}_5$ .

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#### ANALIZE ROENTGENOSTRUCTURALE PRIVIND SISTEMUL $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$

(Rezumat)

Prin analize reontgenostructurale se studiază structura compușilor obținuți prin introducerea, în condiții de stoechiometrie sau deasupra acestor condiții, a CaO în ferita  $\text{NiFe}_2\text{O}_4$ .

Difractogramele (obținute cu un difractometru de radiații X DRON-1) și debyegramele (obținute la o instalație de tipul TUR-M-62) au permis ca din indexarea liniilor spectrale să se determine distanțele interplanare  $d_{hkl}$  și constanța rețelei  $a$ .

Din rezultatele obținute și din variația constantei de rețea funcție de concentrația  $x$  în ioni de Ca, s-au putut determina limitele de solubilitate ale calciului în ferita  $\text{NiFe}_2\text{O}_4$  cu formarea feritei spinelice monofazice  $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$  sau formarea unui compus plurifazic, determinat de segregarea calciului la granițele cristalitelor și formarea unor compuși Ca—Fe.





## SUR LE COMPORTEMENT DES FERROFLUIDES EN PRÉSENCE D'UN CHAMP MAGNÉTIQUE NONHOMOGÈNE TOURNANT

PAR

C. COTAE, GHIORGHE CĂLUGĂRU et EMIL LUCA

**1. Introduction.** Le comportement des ferrofluides (suspensions de grains magnétiques monodomainiques, stables, dans un fluide porteur) en présence d'un champ magnétique tournant constitue un sujet largement étudié dans une série de travaux antérieurs [1]...[15], tant de point de vue théorique [1]...[10], que de point de vue expérimental [1], [11]...[15]. Les conclusions résultées, tant par la considération de divers modèles théoriques proposés que par les vérifications expérimentales mettent en évidence le fait que le ferrofluide est entraîné dans un mouvement de rotation dans le cas d'utilisation d'un champ magnétique tournant avec une distribution spatiale nonuniforme aussi que dans le cas des ferrofluides hétérogènes, placés dans des champs magnétiques tournants uniformes.

Dans l'ouvrage de I. Ia. K a g a n et collaborateurs [11] on met en évidence de point de vue expérimental le changement du sens de rotation du ferrofluide à une certaine valeur du champ magnétique tournant (qui a une distribution spatiale nonuniforme) tandis que dans les travaux [13], [15] on met en évidence le fait que la valeur du champ magnétique d'inversement dépend tant de l'aimantation du ferrofluide que de la viscosité du fluide de base, aux champs forts le ferrofluide présentant un deuxième changement du sens de rotation.

De même, dans l'ouvrage [11] on étudie le comportement d'un cylindre diélectrique placé dans un ferrofluide qui tourne sous l'action d'un champ magnétique tournant et on constate que le sens de rotation du cylindre est indépendant du sens de rotation du ferrofluide.

Dans cet ouvrage on présente le comportement d'un tore diélectrique, qui flotte à la surface d'un ferrofluide nonhomogène, du point de vue de la distribution dimensionnelle des grains en suspension.

On présente l'influence de la concentration de grandes particules ( $> 200 \text{ \AA}$ ) dans la suspension et de la viscosité du fluide de base sur la fréquence de rotation du tore diélectrique.

**2. La technique expérimentale.** On a utilisé trois tores diélectriques en matériel plastique dont les dimensions sont celles du tableau 1.

Tableau 1

| Le tore | L'hauteur<br>$h$<br>mm | Le rayon<br>intérieur<br>$r$<br>mm | Le rayon<br>extérieur<br>$R$<br>mm | $R$ | Le rayon moyen<br>$R_m$<br>mm |
|---------|------------------------|------------------------------------|------------------------------------|-----|-------------------------------|
| 1       | 4,5                    | 7                                  | 9                                  | 2   | 8                             |
| 2       | 4,5                    | 10                                 | 12                                 | 2   | 11                            |
| 3       | 4,0                    | 12                                 | 17                                 | 5   | 14,5                          |

Les échantillons de ferrofluide utilisés contiennent grains de différentes dimensions en diverses concentrations. On a réalisé la suspension des particules en  $\text{Fe}_3\text{O}_4$  dans pétrole ( $\eta = 2,25 \cdot 10^{-3} \text{ Ns/m}^2$ ) et dans huile pour transformateur ( $\eta = 22 \cdot 10^{-3} \text{ Ns/m}^2$ ) stabilisés à l'aide de l'acide oléique. La concentration volumique des grains magnétiques dans ces échantillons est  $\varepsilon = 0,01$ , tandis que l'intensité d'aimantation  $M = 4 \cdot 10^3 \text{ A/m}$ , à un champ de  $96 \cdot 10^3 \text{ A/m}$ .

On a préparé les échantillons en utilisant quatre genres de ferrofluides ( $A, B, C, D$ ) ayant la dimension moyenne des grains magnétiques différente ( $d_A = 200 \text{ \AA}$ ,  $d_B = 600 \text{ \AA}$ ,  $d_C = 800 \text{ \AA}$  et  $d_D = 1000 \text{ \AA}$ ). On a combiné le ferrofluide type  $A$  avec les ferrofluides type  $B, C$  et  $D$  en divers pourcentages (0 % ; 20 % ; 50 % ; 80 % et 100 %). Les échantillons type  $UT-AC$  ont été obtenus en utilisant les échantillons  $A$  et  $C$ ; on a floculé les grains magnétiques existants en pétrole ( $AC$ ) et on les a introduit en huile pour transformateur, réalisant ainsi la variation de la viscosité du fluide de base.

Le tableau 2 décrit les échantillons utilisés dans les expériences.

Le champ magnétique tournant de fréquence 25 Hz a été produit à l'intérieur du stator d'un moteur asynchrone à bobinage triphasique. Il présente une distribution spatiale nonuniforme tant axiale que radiale,

Tableau 2

| L'échantillon<br>$AB$ | $A\%$<br>$d_A$ | $B\%$<br>$d_B$ | L'échantillon<br>$AC$    | $A\%$<br>$d_A$ | $C\%$<br>$d_C$ |
|-----------------------|----------------|----------------|--------------------------|----------------|----------------|
| $A_{100}-B_0$         | 100            | 0              | $A_{100}-C_0$            | 100            | 0              |
| $A_{80}-B_{20}$       | 80             | 20             | $A_{80}-C_{20}$          | 80             | 20             |
| $A_{50}-B_{50}$       | 50             | 50             | $A_{50}-C_{50}$          | 50             | 50             |
| $A_{20}-B_{80}$       | 20             | 80             | $A_{20}-C_{80}$          | 20             | 80             |
| $A_0-B_{100}$         | 0              | 100            | $A_0-C_{100}$            | 0              | 100            |
| L'échantillon<br>$AD$ | $A\%$<br>$d_A$ | $D\%$<br>$d_D$ | L'échantillon<br>$UT-AC$ | $A\%$<br>$d_A$ | $C\%$<br>$d_C$ |
| $A_{100}-D_0$         | 100            | 0              | $A_{100}-C_0$            | 100            | 0              |
| $A_{80}-D_{20}$       | 80             | 20             | $A_{80}-C_{20}$          | 80             | 20             |
| $A_{50}-D_{50}$       | 50             | 50             | $A_{50}-C_{50}$          | 50             | 50             |
| $A_{20}-D_{80}$       | 20             | 80             | $A_{20}-C_{80}$          | 20             | 80             |
| $A_0-D_{100}$         | 0              | 100            | $A_0-C_{100}$            | 0              | 100            |



comme dans l'ouvrage [11]. L'étude de la rotation des tores diélectriques a été fait dans des récipients de 60 mm diamètre remplis de ferrofluide jusqu'à une hauteur de 20 mm, la surface libre du ferrofluide étant placée dans la zone de champ axial maximum.

Pour déterminer la fréquence de rotation des tores dans toutes les expériences on a attendu qu'on établisse la fréquence de rotation stationnaire égale à la fréquence de rotation du ferrofluide.

Par conséquent, les conclusions qui en résultent pour les tores diélectriques peuvent être particularisées aussi pour la couche du ferrofluide qui entraîne ces tores.

**3. Résultats expérimentaux et discussions.** Dans la fig. 1 on représente la dépendance de la fréquence de rotation des trois tores diélectriques en fonction de la concentration de grandes grains, pour une valeur du courant en stator de 4 A (les ferrofluides AC dans la fig. 1 a et UT-AC dans la fig. 1 b). On constate une forte dépendance de la fréquence de rotation du ferrofluide et respectivement des tores diélectriques en fonction de la concentration de grandes grains de la suspension.

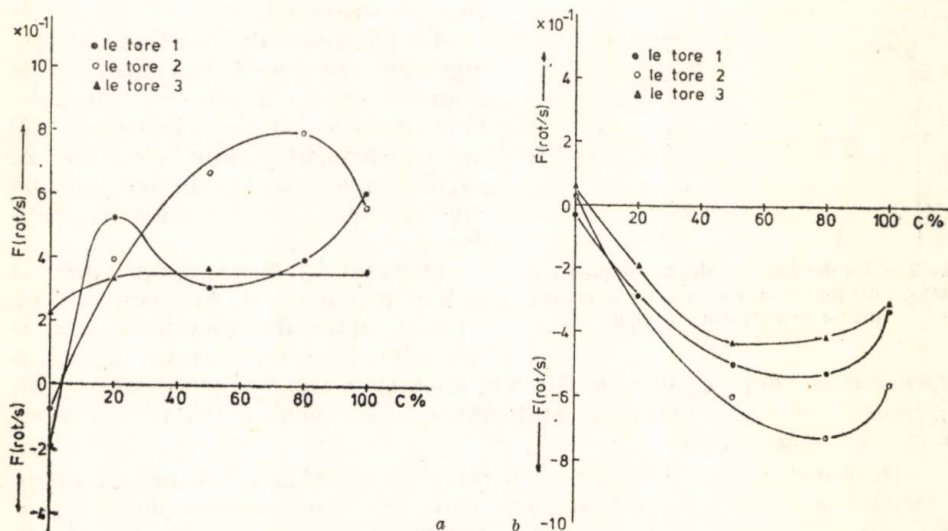


Fig. 1 — La dépendance de la fréquence de rotation en fonction de la concentration des grandes grains ; a — échantillon AC ; b — échantillon UT-AC.

Au fur et à mesure que cette concentration augmente, la fréquence de rotation augmente aussi gagnant une valeur maxima à une concentration en grandes grains de 80%. On constate aussi que la fréquence maxima dépend du rayon moyen du tore rapporté au rayon du récipient contenant le ferrofluide. La fréquence maxima correspond à un rayon moyen du tore d'environ  $1/3$  du rayon moyen du récipient.

La viscosité de la base du ferrofluide a une forte influence sur le comportement des tores diélectriques. Dans le cas du ferrofluide AC, le pétrole étant la base ( $\eta = 2,25 \cdot 10^{-3} \text{ Ns/m}^2$ ), les tores sont entraînés dans un mou-

vement positif de rotation dans le même sens que le champ magnétique, lorsque la concentration des grains  $C$  dépasse 10%.

Dans le cas du ferrofluide  $UT-AC$ , l'huile pour transformateur étant la base ( $\eta = 22 \cdot 10^{-3} \text{ Ns/m}^2$ ), les tores sont entraînés dans un mouve-

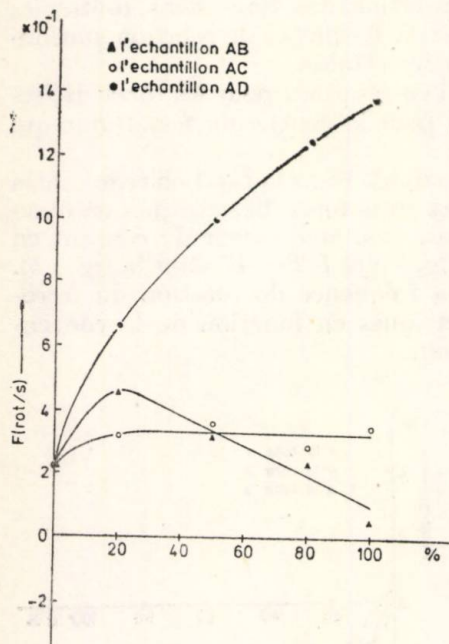


Fig. 2 — La dépendance de la fréquence de rotation du tore 3 en fonction de la concentration des grandes grains.

dimension moyenne  $d_B$  dans la fig. 3 a,  $d_C$  dans la fig. 3 b, et  $d_D$  dans la fig. 3c), pour différents courants statoriques, donc pour différentes valeurs du champ magnétique.

On constate que dans le cas du ferrofluide  $AB$  la valeur maximum de la fréquence de rotation est atteinte pour une concentration de 20 %  $B$  tandis que la valeur minimale pour 100 %  $B$ . Dans le cas du ferrofluide  $AC$  la valeur maximum est atteinte à 100 %  $C$ , tandis que la valeur minimale pour 0 %  $C$ . Dans le cas ci on constate que la valeur de la fréquence à une concentration de 50%  $C$ , dépasse la valeur de toutes fréquences de rotation obtenues dans le cas du ferrofluide  $AB$ , de n'importe quelle concentration. Quant à la dépendance en fonction du courant statorique on constate que pour les ferrofluides  $AB$  et  $AD$  la fréquence maximum est atteinte à un courant de 5 A, tandis que pour le ferrofluide  $AC$  la fréquence maximum est atteinte à un courant de 3 A.

**4. Conclusions.** On étudie le comportement des tores diélectriques ui flottent à la surface d'un ferrofluide nonhomogène, dans un champ magnétique tournant à distribution spatiale nonuniforme.

ment de rotation négatif pour n'importe quelle concentration de grandes grains. On constate aussi un mode de variation de la fréquence de rotation en fonction de la concentration de grandes grains beaucoup plus régulier que dans le cas des ferrofluides ayant le pétrole comme base.

Dans la fig. 2 on représente la dépendance de la fréquence de rotation du tore 3 en fonction de la concentration de grandes grains pour trois valeurs moyennes de grandes grains et pour la même valeur de 4 A du courant en stator.

La fréquence de rotation du tore augmente lorsque la concentration de grandes grains augmente, spécialement dans le cas des échantillons  $AD$  qui contiennent grains de 1 000 Å, gagnant une valeur maxima de 1,38 rot/s à 100 % concentration de grandes grains.

Dans les fig. 3 a, b, c, on représente la dépendance de la fréquence de rotation du tore 3 en fonction de la concentration de grandes grains (ayant la



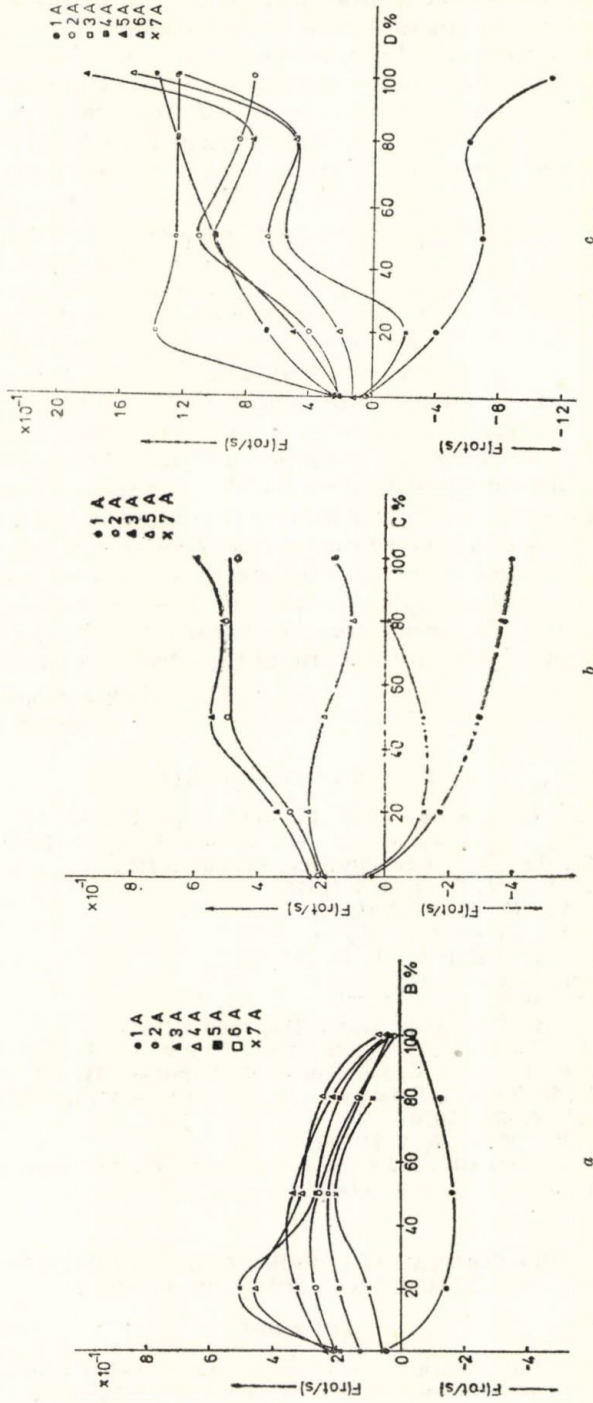


Fig. 8 -- La dépendance de la fréquence de rotation du tore 3 en fonction de la concentration des grandes grains :  
a — échantillon AB ; b — échantillon AC ; c — échantillon AD.

Des observations expérimentales il en résulte qu'un corp diélectrique en forme de tore est entraîné dans un mouvement de rotation dans le même sens que le ferrofluide. Dans ces conditions-ci le ferrofluide présente un double changement du sens de la rotation [15], changement subi aussi par le tore. Ce comportement d'un tore est différent de celui évidencié par Kagan et ses collaborateurs [11] qui utilisent dans leurs expériences un cylindre diélectrique qui tourne tout le temps en sens inverse au champ magnétique pour n'importe quel sens de rotation du ferrofluide.

Nous considérons que ce comportement différent peut être attribué au fait que dans les expériences de Kagan [11] il y a une seule surface, extérieure, du diélectrique en contact avec le ferrofluide, tandis que dans nos expériences il y en a deux, une surface extérieure et une intérieure.

Kagan montre [11] qu'un moteur ayant un rotor diélectrique de forme cylindrique plongé dans un ferrofluide offre la possibilité d'être adapté à divers systèmes de commande automatique. Nous considérons que plus qu'un système à cylindre, un système à tore diélectrique plongé dans un ferrofluide hétérogène (donc qui présente un double changement du sens de rotation) peut offrir des possibilités d'utilisation dans les systèmes d'automatisation beaucoup plus larges (à la condition qu'il ne soit pas nécessaires des couples forts). On peut déterminer la fréquence de rotation et le moment quand survient le changement du sens de rotation d'un tore à certaines dimensions à la condition qu'on connaît la concentration des grains de différentes dimensions en suspension, la viscosité de la base du ferrofluide et la valeur du champ magnétique tournant (la valeur du courant statorique).

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#### ASUPRA COMPORTĂRII FEROFUIDELOR ÎNTR-UN CÎMP MAGNETIC NEOMOGEN ROTITOR

(Rezumat)

Se studiază efectul ferofluidelor neomogene aflate într-un câmp magnetic rotitor, neuniform, asupra unui tor dielectric care plutește la suprafața ferofluidului.





Conf. dr. VIOREL CAMIL MURGESCU

(1929—1979)

În seara zilei de 10 mai 1979 s-a produs decesul fulgerător al conferențiarului doctor *Viorel Camil Murgescu*, distins matematician care a onorat îndelungată vreme învățămîntul din Institutul politehnic Iași, unde a funcționat începînd din 1952, mai întîi în calitate de asistent, apoi din 1962 pînă în 1969 ca lector, iar în urma unui strălucit concurs, în calitate de conferențiar, din 1969 pînă la deces.

Printre disciplinele care le-a predat la diverse facultăți, menționăm: matematici speciale, la Facultatea de electrotehnică, analiză matematică și matematici speciale, la Facultatea de construcții și, la aceeași facultate, la care a funcționat exclusiv din 1968, algebră liniară și geometrie analitică. De asemenea, a condus seminarii la aceleași discipline în cadrul a patru facultăți din Institut.

Ca urmare a acestei fructuoase activități, *Viorel Murgescu* a publicat în cadrul Institutului un excelent Curs de analiză matematică și matematici speciale, în trei volume, destinat studenților facultății de construcții, ultimul volum difuzat imediat după dispariția autorului, epuizîndu-se într-un timp record.

Defunctul proiecta, de asemenea, o reeditare substanțial revăzută și completată a unei culegeri de probleme, publicată tot în cadrul Institutului, în 1960, și editarea unui Curs de algebră liniară și geometrie analitică, din care au rămas în manuscris unele capitole.

După opinia unanimă a colectivului catedrei și a numeroși colegi, precum și a marelui număr de studenți, *Viorel Murgescu* a avut excepționale calități de profesor și educator, impunându-se prin ținută și cunoștințe științifice, evidențiindu-se prin spiritul de dreptate și imparțialitate și dobândind un bine meritat prestigiu.

A contribuit la pregătirea și organizarea olimpiadelor studențești și a condus simpozioane matematice pentru studenții facultăților de construcții și electrotehnică. Alegerea temelor acestor simpozioane reflecta preocuparea sa de a completa pregătirea științifică a viitorilor ingineri. A ținut cursuri de matematici cu caracter postuniversitar, organizate de Facultatea de construcții pentru cadrele didactice, în două serii, începând din 1969.

Ca formație științifică inițială, *Viorel Murgescu* a fost geometru. A continuat bunele tradiții ale Școlii de geometrie din Iași, aducând contribuții valoroase la studiul spațiilor cu conexiune afină, cu torsiune și a spațiilor Weyl generalizate. Importanța acestor cercetări este în necontestată creștere, fapt care a ieșit pregnant în evidență la Simpozionul organizat de Facultatea de matematică a Universității „Al. I. Cuza”, cu ocazia sărbătoririi centenarului nașterii lui Albert Einstein, cu puțin timp înaintea decedului regretatului nostru coleg. Unul din participanți, fizico-matematician din Timișoara, a propus ca spațiile Weyl generalizate să fie denumite „spații Murgescu” și sperăm ca această onoare postumă să fie acceptată de cercurile științifice internaționale.

*Viorel Murgescu* a fost prezent la numeroase manifestări științifice din țară, iar în 1968 a participat la Congresul matematicienilor din Austria, la Linz, cu o comunicare asupra spațiilor Weyl generalizate, al cărei rezumat a fost publicat în actele Congresului.

Mai semnalăm că a fost invitat să țină conferințe în Belgia (1969) și în Italia (1971) și să participe la Colocviul de ecuații funcționale de la Oberwolfach, R.F.G. (1968).

Ca urmare a funcționării în cadrul Institutului politehnic și un timp ca cercetător la Institutul de matematică al Filialei Iași a Academiei R.S.R., o a doua latură a preocupărilor științifice ale lui *Viorel Murgescu* a fost constituită de cercetări în domeniul matematicilor aplicate (metode numerice și operaționale în construcții), de generalizarea unor legi de probabilitate și de unele probleme de analiză matematică (ecuații diferențiale și cu diferențe, ecuații funcționale, funcții speciale).

A publicat și a comunicat un număr de 45 de lucrări.

Lucrările sale au fost integral și favorabil recenzate în revistele internaționale de recenzii ca *Mathematical Reviews*, *Zentralblatt für Mathematik*, *Bulletin Signalétique* și *Referativnyi Jurnal*. De altfel, conducerile primelor două reviste l-au cooptat de multă vreme ca recenzent și în această calitate, a publicat 120 de recenzii.

În direcția integrării învățământului cu cercetarea și producția a colaborat la unele contracte ale catedrelor de specialitate de la Facultatea de construcții și a publicat lucrări în colaborare cu cadre didactice de la această



facultate. Convins de necesitatea și urgența integrării, a subliniat deseori importanța acestei probleme în dezbaterile din catedră.

*Viorel Murgescu* a păstrat toată viața o profundă recunoștință foștilor săi profesori, iar pentru profesorul său de matematică din liceu, Benone Constantinescu, a publicat un amănunțit și frumos necrolog.

Mai menționăm de asemenea activitatea depusă în calitate de membru în unele comisii de examinare pentru admiterea la doctoratură în Institutul politehnic, în comisiile de examinare la matematici din cadrul examenelor pentru doctoranzi și în comisii de concurs pentru ocuparea unor posturi din catedră.

Membrii catedrei îi vor păstra o neștearsă amintire pentru calitățile lui deosebite de om, profesor și om de știință.

*Catedra de matematică*

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Lector ANA TRIANDAF

(1910—1979)

Data de 11 ianuarie a acestui an a însemnat o pierdere dureroasă pentru Catedra de matematici a Institutului politehnic din Iași, deoarece în acea zi a încetat din viață o ființă deosebită, lector *Ana Triandaf*.

S-a stins din viață după o activitate devotată de 26 de ani în slujba învățământului politehnic.

Activitatea în cadrul Institutului politehnic din Iași și-a început-o în anul 1951 în condițiile de loc comode ale unei munci permanente, în care un loc esențial în acea perioadă îl aveau cursurile muncitorești, care s-au întins pe o etapă de aproape patru ani; în acele momente, lector *Ana Triandaf*, în condiții grele, făcea înțeleasă, până noaptea târziu, matematica, unor oameni care munceau și ei din greu și unica răsplată, cea adevărată, au fost succesele în urma unei munci conștiincioase și respectul deplin din partea celor care învățau.

Mai mult de 20 de ani, chiar și atunci când ținea cursuri proprii, lector *Ana Triandaf* a muncit cu pasiune pentru susținerea seminariilor de Analiză matematică la Facultatea de electrotehnică, seminarii permanent îmbunătățite și ținute la ridicat nivel științific. Acest devotament s-a prelungit și după anul 1967, când lector *Ana Triandaf* fiind pensionată, a fost solicitată să-și continue activitatea datorită calităților sale pedagogice și științifice deosebite.

Consultațiile pe care lector *Ana Triandaf* le acorda studenților erau alte momente de înaltă dăruire profesională; ele întreceau cu mult orele fixate și erau oferite cu răbdare și bunătațe atît studenților români cît și celor străini, cu aceștia din urmă efectuîndu-le mult timp și separat.

În calitate de lector, *Ana Triandaf* a predat timp de trei ani cursuri studenților Facultății de electrotehnică, secția serală, și cursuri de Geometrie analitică și diferențială la Facultatea de construcții. Erau cursuri accesibile, riguroase, predate cu pricepere pedagogică și științifică, legate de specificul facultăților.

În urma acestei activități, împreună cu două colege de catedră, lector *Ana Triandaf* a scris o Culegere de probleme de geometrie analitică și diferențială, publicată în anul 1973 în Editura didactică și pedagogică, București, carte care conține exerciții și probleme interesante, accesibile, judicioase alese, prezentate gradat, acoperind întreg materialul și care are meritul deosebit al folosirii învățămîntului programat, aspect rar întîlnit în culegeri de acest fel.

Înainte de a exista posibilitatea încheierii de contracte cu întreprinderi industriale, lector *Ana Triandaf* a studiat unele probleme tehnice cu ajutorul matematicii, lucrînd la teme aplicative concrete care foloseau programarea liniară și statistica, concretizate în lucrări științifice publicate.

*Ana Triandaf* s-a născut la 6 martie 1910 la Adjud. A făcut școala primară la Iași și a urmat liceul Oltea-Doamna din Iași. Obține licența în anul 1937. Din 1937 pînă în 1951, cînd își începe activitatea la Institutul politehnic din Iași, a fost profesor de matematică la diferite școli din Iași, printre care liceele „Eminescu”, „Sadoveanu” și „Alecsandri”.

Colega noastră care a dispărut a fost un exemplu de modestie și atașament față de colectivul catedrei; priceperea ei a fost de multe ori elementul determinant al unor reușite reuniuni colegiale; momentele grele ale vieții sale le-a depășit de cele mai multe ori singură și totdeauna a fost discretă și apropiată de colegi.

De aceea va rămîne un exemplu viu de devotament în muncă, de modestie și atitudine morală demnă în viață.

*Catedra de matematică*